SOLUTIONS MANUAL

CONTINUOUS AND DISCRETE SIGNALS AND SYSTEMS

SECOND EDITION

Samir S. Soliman
Mandyam D. Srinath

PRENTICE HALL Upper Saddle River, NJ 07458

Associate Editor: Alice Dworkin
Production Editor: Mindy De Palma
Supplement Cover Designer: Liz Nemeth
Special Projects Manager: Barbara A. Murray
Supplement Cover Manager: Paul Gourhan
Manufacturing Buyer: Donna Sullivan

A Pearson Education Company
Upper Saddle River, NJ 07458

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-774308-4

Prentice-Hall International (UK) Limited,London
Prentice-Hall of Australia Pty. Limited, Sydney
Prentice-Hall Canada Inc., Toronto
Prentice-Hall Hispanoamericana, S.A., Mexico
Prentice-Hall of India Private Limited, New Delhi
Prentice-Hall of Japan, Inc., Tokyo
Pearson Education Asia Pte. Ltd., Singapore
Editora Prentice-Hall do Brasil, Ltda., Rio de Janeiro

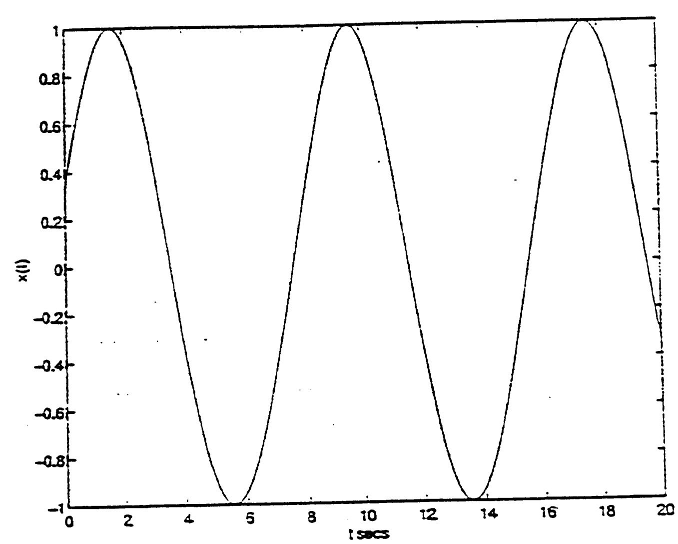
CONTENTS

Chapter 1

1.1

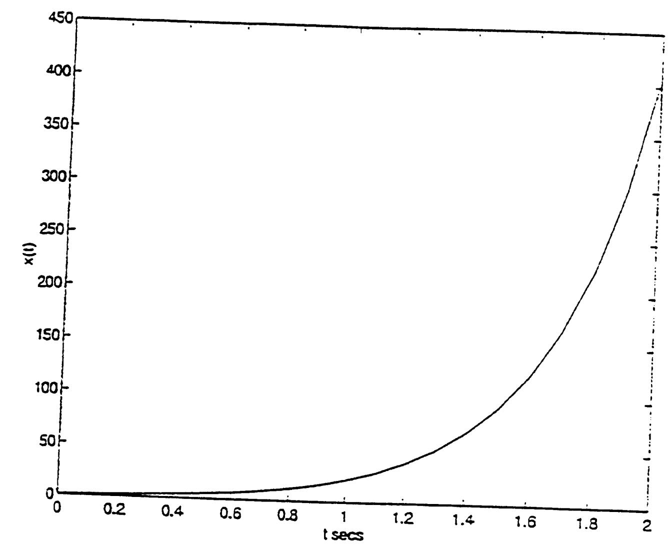
Signal	Period
$\cos(\pi t)$	2
$\sin(2\pi t)$	1
$\cos(3\pi t)$	$\frac{1}{3}$
$\sin(4\pi t)$	$\frac{1}{2}$
$\cos(\frac{\pi}{2}t)$	4
$\sin(\frac{\pi}{3}t)$	6
$\cos(\frac{5\pi}{2}t)$	$\frac{4}{5}$
$\sin(\frac{4\pi}{3}t)$	$\frac{3}{2}$
$\cos(\frac{\pi}{4}t)$	8
$\sin(\frac{2\pi}{3}t)$	$\frac{1}{3}$
$\cos(\frac{3\pi}{5}t)$	$\frac{10}{3}$

1.2

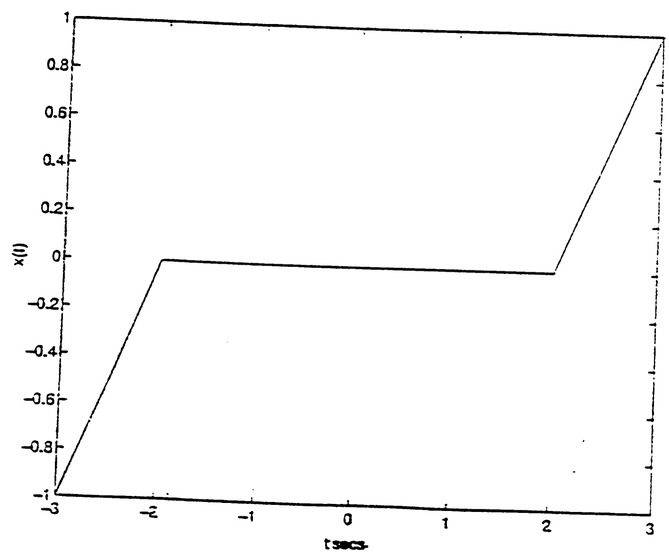


(a) $x(t) = \sin(\frac{\pi}{4}t + 20°)$

1.2

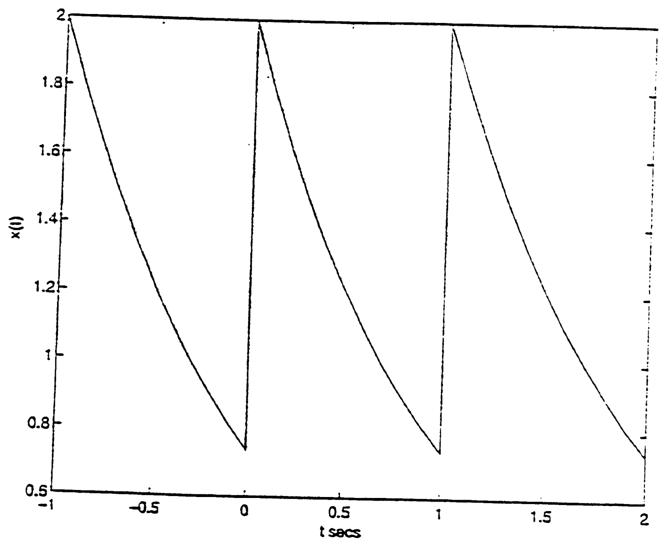


(b) $x(t) = t + e^{3t}$



$$\text{(c) } x(t) = \begin{cases} t+2 & t \leq -2 \\ 0 & -2 \leq t \leq 2 \\ t-2 & 2 \leq t \end{cases}$$

1.2



(d) $x(t) = 2\exp[-t]$, $0 \le t < 1$, and $x(t+1) = x(t)$

1.3 $x(t+T) = x(t)$ for every t.

if we substitute $t = t+T$

$$x(t+2T) = x(t+T) = x(t)$$

i.e $x(t)$ is periodic with period $2T$

if we substitute $t = t+2T$

$$x(t+3T) = x(t+2T) = x(t)$$

i.e $x(t)$ is periodic with period $3T$

In general $x(t)$ is periodic with period nT

1.4 $x_3(t) = a x_1(t) + b x_2(t)$

$$x_3(t+T) = a x_1(t+T) + b x_2(t+T)$$

$$= a x_1(t) + b(x_2(t)$$

$$= x_3(t)$$

1.5 (a) $\sin \frac{2\pi}{3} t$ is periodic with period 3 while $\sin \frac{16\pi t}{3}$ is peridic with period $\frac{3}{8}$ Then $x(t)$ is periodic with period 3.

1.5 (b) $\exp[j\frac{2\pi}{6}t]$ is periodic with period 6 and

$\exp[-j\pi t]$ is periodic with period 2.

Then $x(t)$ is periodic with period 6.

(c) $\exp[j\frac{2\pi}{5}t]$ is periodic 5 while

$\exp[j3t]$ is periodic with period $\frac{2\pi}{3}$

Since the ratio of these two periods, $\frac{5}{\frac{2\pi}{3}} = \frac{15}{2\pi}$

is not a rational number, $x(t)$ is not periodic.

1.6 (a) $T_1 = 6, T_2 = \frac{3}{4}, \frac{T_1}{T_2} = 8$. Periodic, with period $T = 6$.

(b) $T_1 = \frac{7}{6}, T_2 = \frac{5}{6}, \frac{T_1}{T_2} = \frac{7}{5}$. Periodic, with period $T = 35$.

(d) $\exp[\frac{5}{6}t]$ is not periodic, so that $x(t)$ is not periodic.

(e) $T_1 = \frac{16}{3}, T_2 = \frac{8\pi}{3}, \frac{T_1}{T_2} = \frac{8\pi}{3}$ is not a rational number. $x(t)$ is not periodic.

1.7 $\exp[j\omega t] = \cos\omega t + j\sin\omega t$ (Euler's formula)

Since $\cos\omega t$ and $\sin\omega t$ are periodic with period $2\pi/\omega$, then their linear combination is also periodic with period $2\pi/\omega$.

1.8 Since $x(t)$ is periodic with period T, then

$$x(at) = x(at+T)$$
$$= x(a(t+T/a))$$

For $x(at)$ to be periodic with period T_1, one needs $T_1 = T/a$

Similarly,

$$x(t/b) = x(t/b + T)$$
$$= x(\tfrac{1}{b}(t+bT))$$

For $x(t/b)$ to be periodic with period T_2, one needs

$$T_2 = bT$$

For $\sin t$, $T = 2\pi$

For $\sin 2t$, $T_1 = \pi = T/2$

For $\sin t/2$ $T_2 = 4\pi = 2T$

1.9 (a)

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} t^2 \sin^2(\frac{\pi}{3}t)dt$$

$$= \lim_{T\to\infty} \frac{1}{4T}\int_{-T}^{T} t^2(1-\cos(\frac{2\pi}{3}t))dt$$

$$= \lim_{T\to\infty} \frac{1}{4T}\left[\frac{2T^3}{3} - \int_{-T}^{T}\cos(\frac{2\pi}{3}t)dt\right] \longrightarrow \infty$$

so that $x(t)$ is neither an energy nor a power signal.

(b)

$$E = \lim_{T\to\infty}\int_{-T}^{T} \exp[-4|t|]\sin^2(\pi t)dt$$

$$= \lim_{T\to\infty}\int_{0}^{T} \exp[-4t][1-\cos(2\pi t)]dt = \frac{\pi^2}{\pi^2+4}$$

so that $x(t)$ is an energy signal with $P = 0$.

(c)
$$P = \lim \frac{1}{2T} \int_{-T}^{T} \exp(8|t|)dt = \lim \frac{e^{8T}-1}{T} \longrightarrow \infty$$

so that $x(t)$ is neither an energy nor a power signal.

(d) $x(t)$ is periodic with period $T = 12/5$s, so that it is a power signal with

$$P = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_0^T |\exp[j\frac{5\pi}{6}t|^2 dt = 1$$

(e) $x(t) = \frac{1}{2}[\sin(\frac{25\pi}{24}t) - \sin(\frac{7\pi}{24}t)]$ and is periodic with $T = 48$ secs. Thus

$$P = \frac{1}{4T}\int_0^T [\sin(\frac{25\pi}{24}t) - \sin(\frac{7\pi}{24}t)]^2 dt$$

$$= \frac{1}{4T}\int_0^T \sin^2(\frac{25\pi}{24}t)dt + \frac{1}{4T}\int_0^T \sin^2(\frac{7\pi}{24}t)]dt - \frac{1}{2T}\int_0^T \sin(\frac{25\pi}{24}t)\sin(\frac{7\pi}{24}t)dt$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

(f)
$$P = \lim_{T\to\infty} \frac{1}{2T}[\int_{-T}^{0} 1dt + \int_0^T e^{-6t}dt]$$

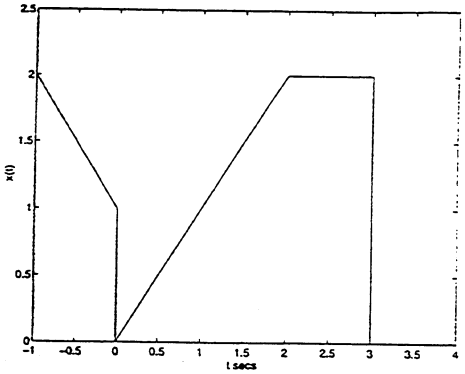
$$= \lim_{T\to\infty} \frac{1}{2T}[T + \frac{1-e^{-6t}}{6}] = \frac{1}{2}$$

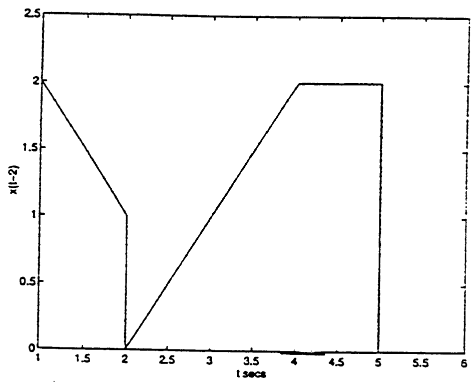
so that $x(t)$ is a power signal.

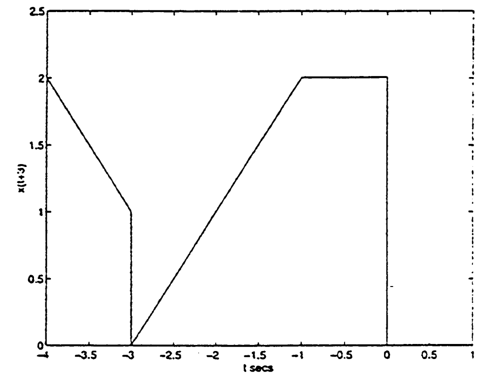
1.10 $\left|\int_0^T x(t)\,dt\right|^2 \le \int_0^T |x(t)|^2 dt \cdot PT$

$\left|\int_0^T x(t)\,dt\right| \le \sqrt{PT}$

1.11

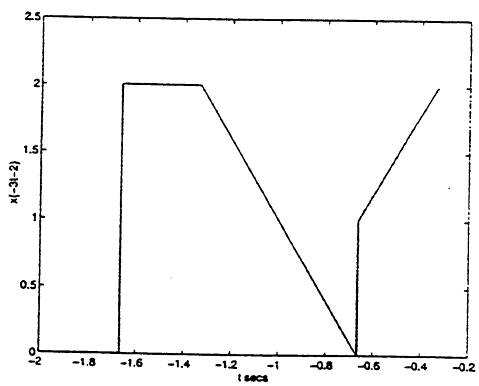


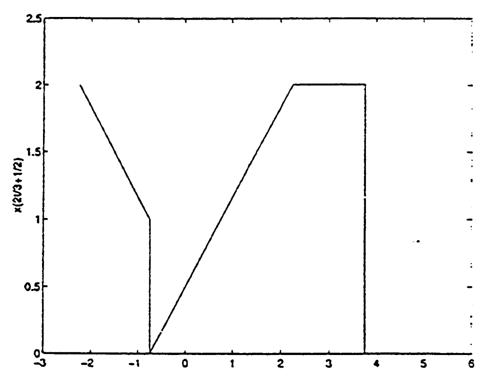




$$x(t-2)=\begin{cases} -t+3 & 1\le t<2 \\ t-2 & 2\le t<2 \\ 2 & 4\le t<5 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t+3)=\begin{cases} -t-3 & -4\le t<-3 \\ t+3 & -3\le t<-1 \\ 2 & -1\le t<0 \\ 0 & \text{otherwise} \end{cases}$$

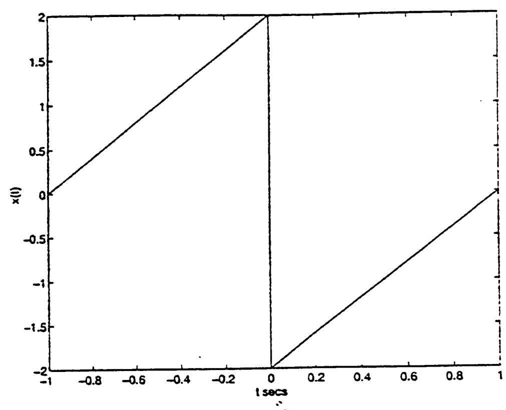


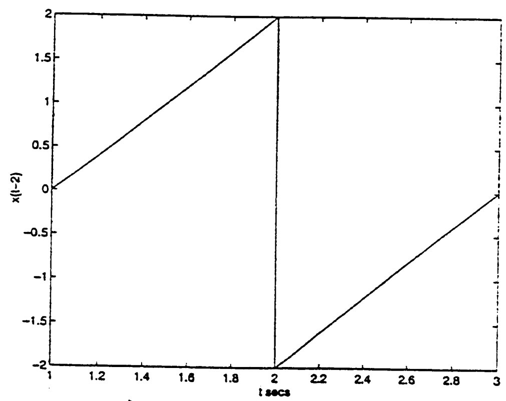


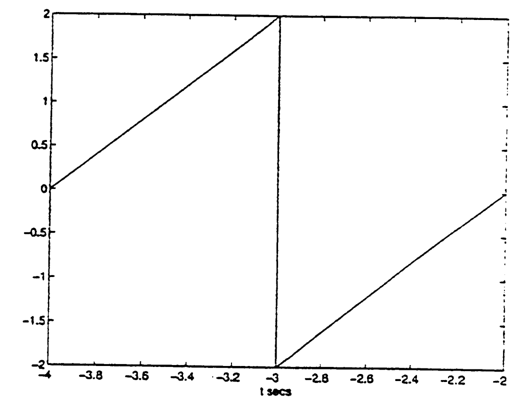
$$x(-3t-2)=\begin{cases} 3t+5 & -\frac{2}{3}<t\le-\frac{1}{3} \\ -3t-2 & -\frac{4}{3}<t\le-\frac{2}{3} \\ 2 & -\frac{5}{3}<t\le-\frac{4}{3} \\ 0 & \\ & \text{otherwise} \end{cases}$$

$$x(\frac{2}{3}t+\frac{1}{2})=\begin{cases} \frac{2}{3}t+\frac{1}{2} & -\frac{9}{4}\le t\leftarrow\frac{3}{4} \\ \frac{2}{3}t+\frac{1}{2} & -\frac{3}{4}\le t<\frac{9}{4} \\ 2 & \frac{9}{4}\le t<\frac{15}{4} \\ 0 & \text{otherwise} \end{cases}$$

1.12

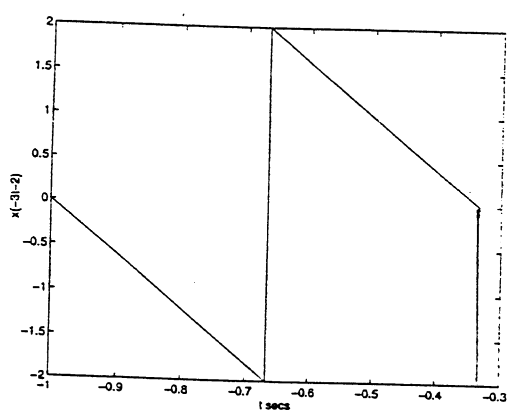


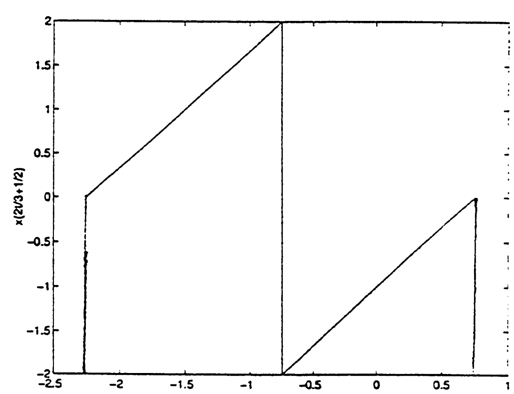




$$x(t-2)=\begin{cases} 2t-2 & 1\le t<2 \\ 2t-6 & 2\le t<3 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t+3)=\begin{cases} 2t+8 & -4\le t<-3 \\ 2t+4 & -3\le t<-2 \\ 0 & \text{otherwise} \end{cases}$$

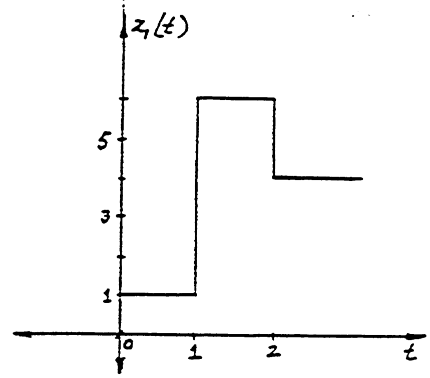




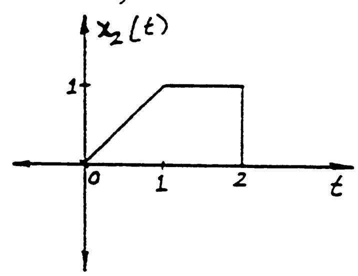
$$x(-3t-2)=\begin{cases} -6t-6 & -1<t\le -\frac{2}{3} \\ -6t-2 & -\frac{2}{3}<t\le -\frac{1}{3} \\ 0 & \\ & \text{otherwise} \end{cases}$$

$$x(\frac{2}{3}t+\frac{1}{2})=\begin{cases} \frac{4}{3}t+3 & -\frac{9}{4}\le t\leftarrow \frac{3}{4} \\ \frac{4}{3}t-1 & -\frac{3}{4}\le t<\frac{3}{4} \\ 2 & \\ & \text{otherwise} \\ 0 & \end{cases}$$

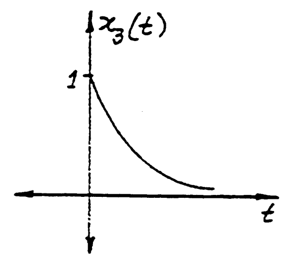
1.13 (a) $x_1(t) = u(t) + 5u(t-1) - 2u(t-2)$



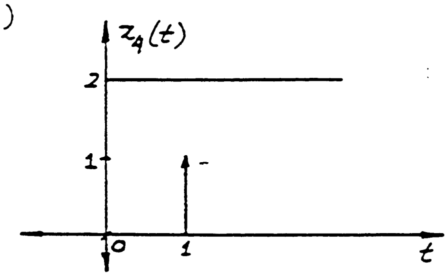
1.13 (b) $x_2(t) = r(t) - r(t-1) - u(t-2)$



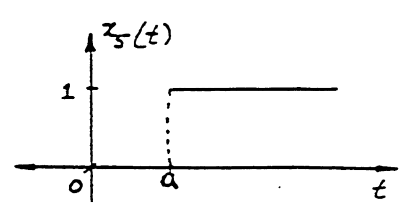
(c) $x_3(t) - e^{-t}u(t)$



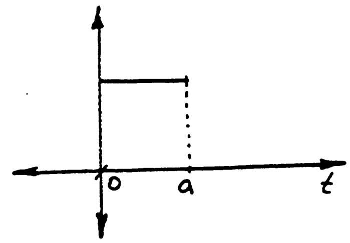
(d) $x_4(t) = 2u(t) + \delta(t-1)$



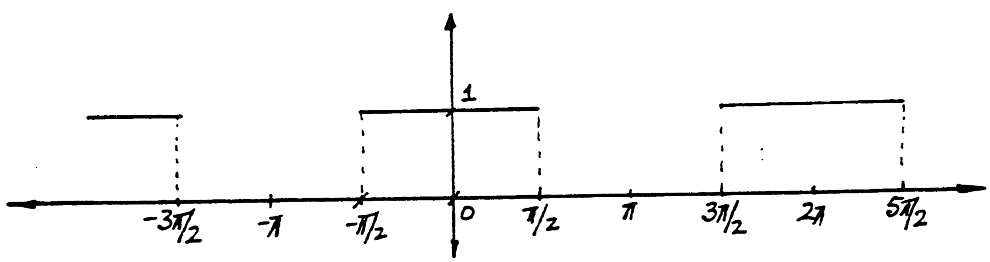
(e) $x_5(t) = u(t)\,u(t-a)$, $a > 0$



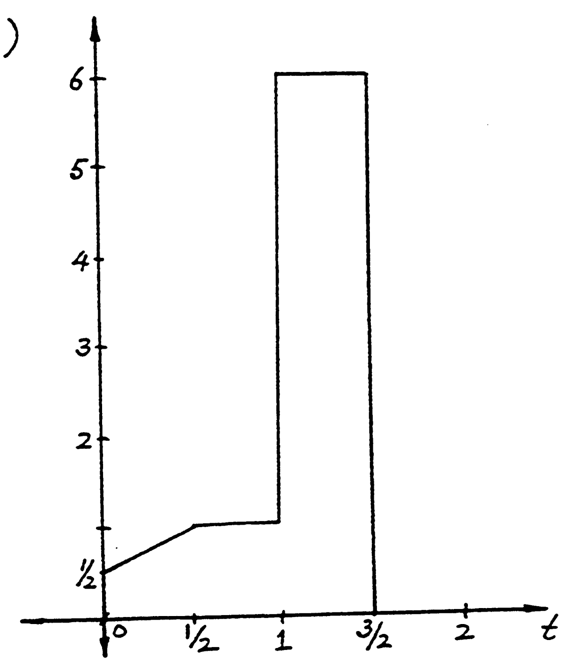
(f) $x_6(t) = u(t)\,u(a-t)$, $a > 0$



(g) $x_7(t) = u(\cos t) = \begin{cases} 1 & \cos t \geq 0 \\ 0 & o.w \end{cases}$

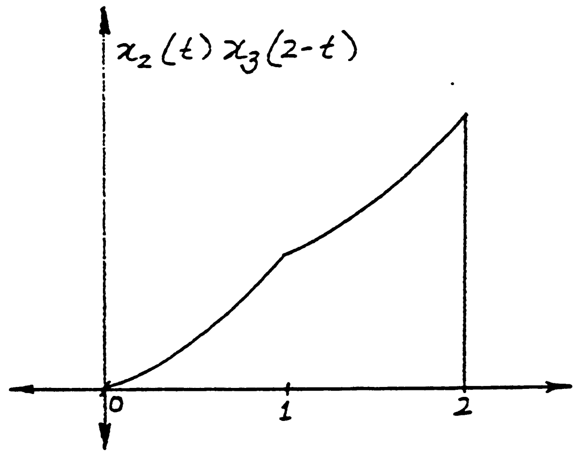


(h) $x_1(t)\; x_2(t + \frac{1}{2})$



(i) $x_1\left(-\frac{t}{3}+\frac{1}{2}\right) x_3(t-2) = 0$ for all t

(j)



1.14 (a) $x_e(-t) = \frac{1}{2}\left[x(-t) + x(t)\right]$

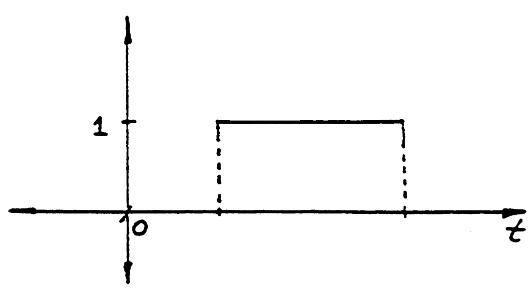
$= x_e(t)$

(b) $x_o(-t) = \frac{1}{2}\left[x(-t) - x(t)\right]$

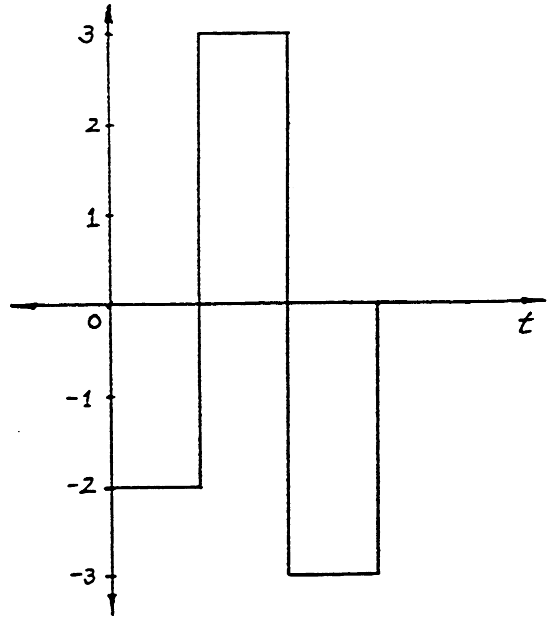
$= -\frac{1}{2}\left[x(t) - x(-t)\right]$

$= -x_o(t)$

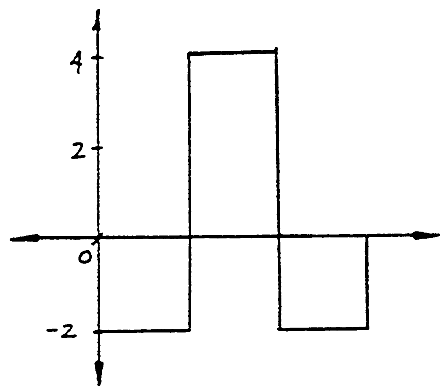
1.15 (a) (i) L+R



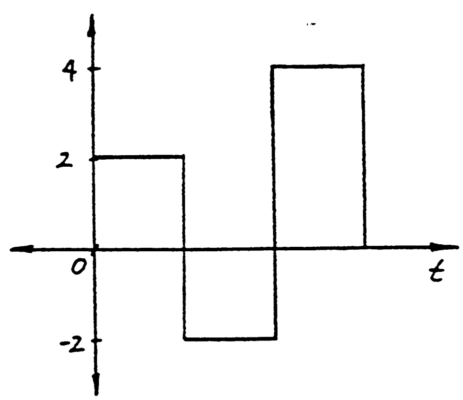
(ii) $L-R$



(b) $(L+R)+(L-R) = 2L$



(c) $(L+R)+(-(L-R)) = (L+R)+(R-L) = 2R$



1.16 $x_1(t) = u(t) - u(t-a)$

$x_2(t) = r(t) - r(t-a) - a\,u(t-b)$

$x_3(t) = \frac{a}{b-a}\left[r(t+b) - r(t+a) - r(t-a) + r(t-b)\right]$

$x_4(t) = (r(t) - r(-t))\,(u(t+a) - u(t-a))$

$x_5(t) = \frac{1}{c}r(t) - \frac{a}{c(a-c)}r(t-c) + \frac{1}{c}r(t-2a+c)$

$x_6(t) = \frac{1}{b-a}\left[r(t+b) - r(t+a)\right] + u(t) - \frac{2}{b-a}\left[r(t-a) - r(t-b)\right]$

1.17 (a) $x_1(t) = A e^{-t/T} u(t)$

$x_1(0) = A \qquad x_1(t') = \frac{A}{e} = A e^{-t'/T}$

$\therefore\ t' = T$

The duration of $x_1(t) = T$

(b) $x_2(t) = x_1(3t) = A e^{-3t/T} u(3t) = A e^{-3t/T} u(t)$

$x_2(t') = \frac{A}{e} = A e^{-3t'/T} \Rightarrow t' = \frac{T}{3}$

The duration of $x_2(t) = \frac{T}{3}$

(c) $x_3(t) = x_1(t/2) = A e^{-t/2T} u(t/2) = A e^{-t/2T} u(t)$

$x_3(0) = A$

$x_3(t') = \frac{A}{e} = A e^{-t'/2T} \Rightarrow t' = 2T$

The duration of $x_3(t) = 2T$

(d) $x_4(t) = 2x_1(t) = 2Ae^{-t/T}\, u(t)$

$x_4(0) = 2A$

$x_4(t') = \frac{2A}{e} = 2Ae^{-t'/T} \Rightarrow t' = T$

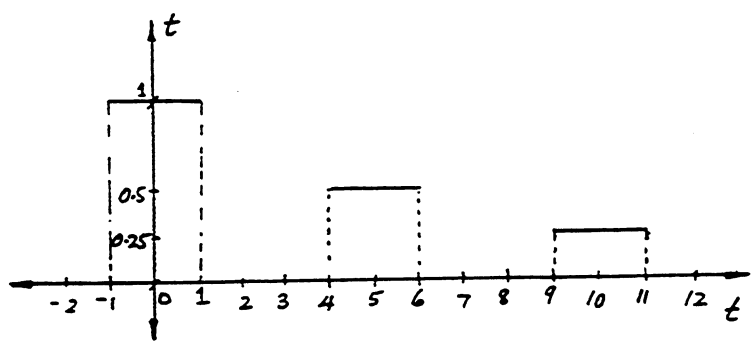
The duration of $x_4(t) = T$

1.18 $x(t) = \text{rect}(t/2)$

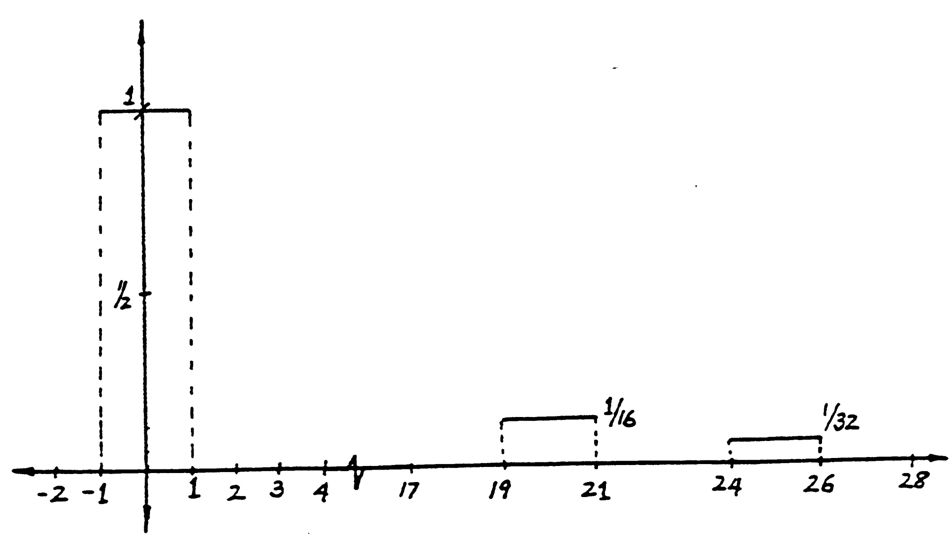
$y(t) = x(t) + 0.5\,x(t - T/2) + 0.25\,x(t - T) \qquad T \gg 2$

(a) $T = 10$

$y(t) = x(t) + 0.5\,x(t-5) + 0.25\,x(t-10) \qquad T \gg 2$



(b) $T = 10$, $z(t) = y(t) - \frac{1}{2}y(t-5) + \frac{1}{8}y(t-15)$



1.19 (a) $P_1(t) = \lim_{\epsilon \to 0} \dfrac{1}{\epsilon \pi \cosh t/\epsilon}$

note that $P_1(t)$ is an even symmetric function with

$$P_1(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon \pi} \to \infty$$

for $t \neq 0$

$$P_1(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon \pi \left(\dfrac{e^{t/\epsilon} + e^{-t/\epsilon}}{2} \right)}$$

$$= \lim_{\epsilon \to 0} \frac{2}{\epsilon \pi (e^{t/\epsilon} + e^{-t/\epsilon})}$$

$$= \lim_{\epsilon \to 0} \frac{2/\epsilon}{\pi (e^{t/\epsilon} + e^{-t/\epsilon})}$$

$$= \lim_{\epsilon \to 0} \frac{-\frac{2}{\epsilon^2}}{\pi (e^{t/\epsilon} + e^{-t/\epsilon})(-\frac{1}{\epsilon^2})}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi (e^{t/\epsilon} + e^{-t/\epsilon})} = 0$$

$$\int_{-\infty}^{\infty} P_1(t)\, dt = \lim_{\epsilon \to 0} \frac{1}{\epsilon \pi} \int_{-\infty}^{\infty} \frac{1}{\cosh (t/\epsilon)}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon \pi} \int_{-\infty}^{\infty} \operatorname{sech} (t/\epsilon)\, dt$$

$t' = \dfrac{t}{\epsilon}$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{sech} (t')\, dt'$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \tan^{-1} (\sinh (t')) \Big|_{-\infty}^{\infty}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 1$$

$\therefore$ $P_1(t)$ can be used as a mathematical model for the delta function

$$\int_{-\infty}^{\infty} P_3(t)\,dt$$

$$= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + 4\pi^2 t^2}\,dt = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+\left(\frac{2\pi t}{\epsilon}\right)^2}\, d\left(\frac{2\pi t}{\epsilon}\right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \tan^{-1}\left(\frac{2\pi t}{\epsilon}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$\therefore$ $P_3(t)$ can be modelled as a delta function

(d) $P_4(t) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2+\epsilon^2}$

note that $P_4(t)$ is an even symmetric function with

$$P_4(0) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi\epsilon^2} = \infty$$

for $t \neq 0$

$$P_4(t) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi(t^2+\epsilon^2)} = 0$$

$$\int_{-\infty}^{\infty} P_4(t)\,dt = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\epsilon}{\pi(t^2+\epsilon^2)}\,dt$$

$$= \frac{1}{\pi} \tan^{-1} \frac{t}{\epsilon}\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$\therefore$ $P_4(t)$ can be used as a model for the delta function.

(e) $P_5(t) = \lim_{\epsilon \to 0} \epsilon \exp[-\epsilon / |t|]$

note that $P_5(t)$ is an even symmetric function with

$$P_5(0) = \lim_{\epsilon \to 0} \epsilon = 0$$

$P_5(t)$ cannot be used as a model for the delta function.

(f) $P_6(t) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin \epsilon t}{t}$

note that $P_6(t)$ is an even symmetric function with

$$P_6(0) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \frac{\sin \epsilon t}{\epsilon t} = 0$$

$P_6(t)$ cannot be used as a model for the delta function.

1.20 (a) $\int_{-\infty}^{\infty} (\frac{2}{3}t - \frac{3}{2})\delta(t-1)dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$

(b) $\int_{-\infty}^{\infty} (t-1)\delta(\frac{2}{3}t - \frac{3}{2})dt = \int_{-\infty}^{\infty} (t-1)\delta(\frac{2}{3}t - \frac{3}{2})dt = \int_{-\infty}^{\infty} (t-1)\frac{3}{2}\delta(t-1)dt = 0$

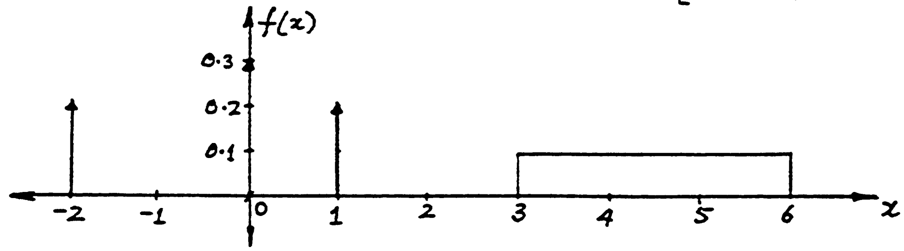
(c) $\int_{-3}^{-2} [\exp(-t+1) + \sin(\frac{2\pi}{3}t)]\delta(t - \frac{3}{2})dt = 0$

(d) $\int_{-3}^{2} [\exp(-t+1) + \sin(\frac{2\pi}{3}t)]\delta(t - \frac{3}{2})dt = e^{-0.5} + \sin(\pi) = e^{-0.5}$

(e) $\int_{-\infty}^{\infty} \exp[-5t+1]\delta'(t-5)dt = -5e^{-5(5)} = -5e^{-25}$

1.21 $P[X \le \alpha] = \int_{-\infty}^{\alpha^+} f(x)\,dx$

$$f(x) = 0.2\,\delta(x+2) + 0.3\,\delta(x) + 0.2\,\delta(x-1) + 0.1\,[u(x-3) - u(x-6)]$$



(a) $P[X \le -3] = \int_{-\infty}^{-3} f(x)\,dx = 0$

(b) $P[X \le 1.5] = \int_{-\infty}^{1.5} f(x)\,dx = \int_{-\infty}^{1.5} [0.2\,\delta(x+2) + 0.3\,\delta(x) + 0.2\,\delta(x-1)]\,dx$

$$= 0.2 + 0.3 + 0.2 = 0.7$$

(c) $P[X \le 4] = \int_{-\infty}^{4} f(x)\,dx$

$$= 0.2 + 0.3 + 0.2 + 0.1 \int_{3}^{4} dx$$

$$= 0.2 + 0.3 + 0.2 + 0.1 = 0.8$$

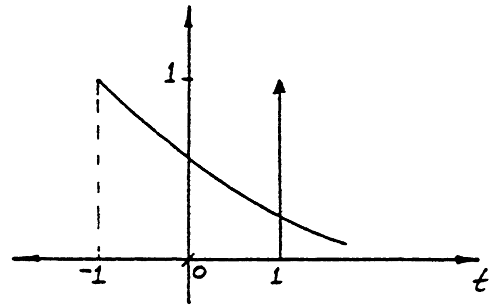
(d) $P[X \le 6] = \int_{-\infty}^{6} f(x)\,dx$

$$= 0.2 + 0.3 + 0.2 + 0.1 \int_{3}^{6} dx$$

$$= 0.2 + 0.3 + 0.2 + 0.3 = 1$$

1.22 $V(t) = e^{-(t+1)} u(t+1) + \delta(t-1)$

(a)



(b) $f(t) = m \frac{d}{dt}[v(t)]$

$$= \frac{d}{dt}\left[\exp[-(t+1)]\, u(t+1) + \delta(t-1)\right] \times 10^{-3}$$

$$= \Big(\exp[-(t+1)]\, \delta(t+1) - \exp[-(t+1)]\, u(t+1)$$

$$+ \delta'(t-1)\Big) \times 10^{-3}$$

$$= (\delta(t+1) - e^{-(t+1)} u(t+1) + \delta'(t-1) \times 10^{-3} \quad N$$

(c) $f_k(t) = k \int_{-\infty}^{t} u(\tau)\, d\tau$

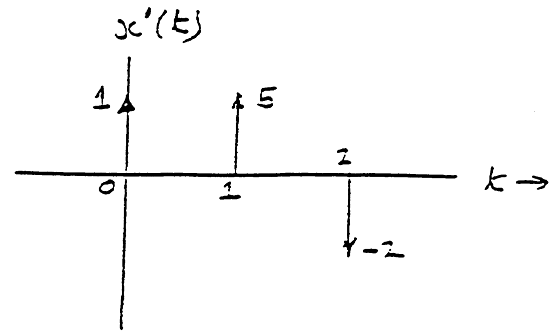
$$= k \int_{-\infty}^{t} \exp[-(\tau+1)]\, u(\tau+1) + \delta(\tau-1)\, d\tau$$

(1) $t < -1$ $\quad f_k(t) = 0$

(2) $-1 < t < 1$ $\quad f_k(t) = 1 - \exp[-(t+1)]$ N

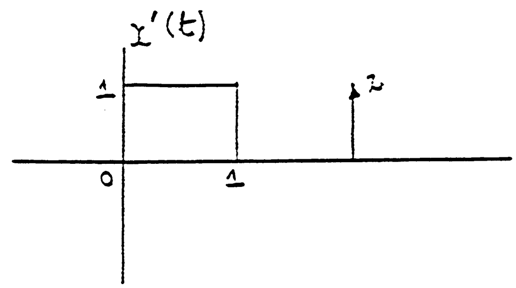
(3) $t > 1$ $\quad f_k(t) = 1 - \exp[-(t+1)] + u(t-1)$ N

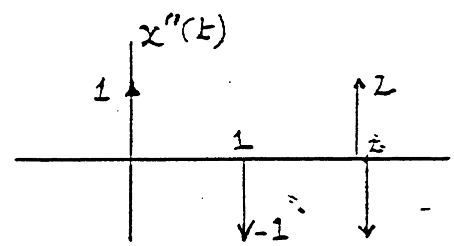
1.23 (a) $x'(t) = \delta(t) + 5\delta(t-1) - 2\delta(t-2)$



(b) $x'(t) = u(t) - u(t-1) + 2\delta(t-2)$

$x''(t) = \delta(t) - \delta(t-1) + 2\delta'(t-2)$

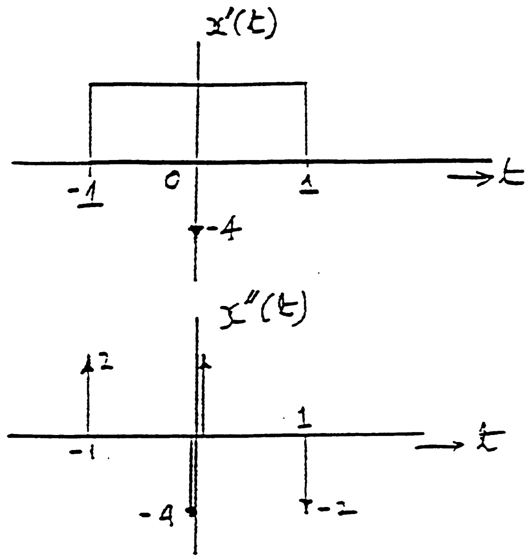




(c) From Equation (1.12), we see that

$x'(t) = 2u(t+1) - 4\delta(t) - 2u(t-1)$

$x''(t) = 2\delta(t+1) - 4\delta'(t) - 2\delta(t-1)$



Chapter 2

2.1 (a) Nonlinear, causal, time-invariant, memoryless.

(b) Nonlinear, causal, time-invariant, memoryless.

(c) Linear, causal, time-invariant, memoryless.

(d) Linear, causal, time-varying, memoryless.

(e) Linear, causal, memoryless.

To check for time-invariance, let $x(t) = u(t)$ and let $x_1(t) = x(t+1)$. Then

$$y(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y_1(t) = \begin{cases} x(t+1) & t \geq 0 \\ -x(t+1) & t < 0 \end{cases}$$

$$= \begin{cases} 1 & t \geq 0 \\ -1 & -1 \leq t < 0 \end{cases} \neq y(t+1)$$

so that the system is time-varying.

(f) Linear, causal, with memory.

To check for time-invariance, let $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau = \int_{-\infty}^{t} x(\tau - t_0)d\tau$$

$$= \int_{-\infty}^{t-t_0} x(\tau)d\tau = y(t - t_0)$$

so that the system is time-invariant.

(g) Linear, causal, time-invariant, with memory.

To check for time-invariance, let $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \int_{0}^{t} x_1(\tau)d\tau = \int_{0}^{t} x(\tau - t_0)d\tau$$

$$= \int_{-t_0}^{t-t_0} x(\tau)d\tau$$

But

$$y(t-t_0) = \int_0^{t-t_0} x(\tau)d\tau \neq y_1(t)$$

so that the system is time-varying.

(h) Linear, causal, time-invariant, with memory.

(i) Nonlinear, time-invariant, causal, memoryless.

(j) Nonlinear, time-invariant, causal, with memory.

(k) Linear, non-causal, time-varying, with memory.

(l) Linear if zero initial conditions, time-invariant, causal, with memory.

2.2 $$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \lim_{\Delta\to 0} \int_{-\infty}^{\infty} x(\tau)\,\frac{1}{\Delta}\,\mathrm{rect}((t-\tau)/\Delta)\,d\tau$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{t-\frac{\Delta}{2}}^{t+\frac{\Delta}{2}} x(\tau)\,d\tau$$

The integral is the area under $x(t)$ between $t-\frac{\Delta}{2}$ and $t+\frac{\Delta}{2}$ This area is approximated by $x(t)\Delta$, therefore

$$\int x(\tau)\delta(t-\tau)\,d\tau = \lim_{\Delta\to 0} \frac{1}{\Delta} x(t)\Delta = x(t)$$

2.3 Note that $u(t+a) * u(t+b) = (t+b+a)\,u(t+b+a)$

(a) $$y(t) = \begin{cases} 1 & \frac{a}{2}+b < t < \frac{3a}{2}+b \\ 0 & \text{o.w.} \end{cases} = \mathrm{rect}\left(\frac{t-a-b}{a}\right)$$

(b) $y(t) = [u(t+\frac{a}{2}) - u(t-\frac{a}{2})] * [u(t+\frac{a}{2}) - u(t-\frac{a}{2})]$

$= (t+a)u(t+a) - 2t\,u(t) + (t-a)u(t-a)$

$$= \begin{cases} t+a & -a \le t \le 0 \\ -t+a & 0 \le t \le a \\ 0 & \text{o.w.} \end{cases}$$

(c) $y(t) = \text{rect}(\frac{t}{a}) * u(t) = [u(t+\frac{a}{2}) - u(t-\frac{a}{2})] * u(t)$

$$= (t+\frac{a}{2})u(t+\frac{a}{2}) - (t-\frac{a}{2})u(t-\frac{a}{2}) = \begin{cases} t+\frac{a}{2} & -\frac{a}{2} \le t \le \frac{a}{2} \\ a & t > a/2 \\ 0 & t < -\frac{a}{2} \end{cases}$$

(d) $y(t) = \text{rect}(\frac{t}{a}) * \text{sgn}\, t = \text{rect}(\frac{t}{a}) * [-u(-t) + u(t)]$

Now $x(t) * u(-t) = \int_t^\infty x(\tau)\,d\tau$

$\therefore y(t) = a$ all t.

(e) $y(t) = u(t) * u(t) = t\,u(t) = r(t)$

(f) $y(t) = t[u(t) - u(t-1)] * u(t)$

$= t\,u(t) * u(t) - (t-1)u(t-1) * u(t) - u(t-1) * u(t)$

$= \frac{t^2}{2}u(t) - \frac{(t-1)^2}{2}u(t-1) - (t-1)u(t-1)$

(g) $y(t) = \text{rect}(\frac{t}{a}) * r(t)$

$$= \begin{cases} 0 & t \le -\frac{a}{2} \\ \int_{t-a/2}^{t+a/2} \tau\, d\tau & -\frac{a}{2} \le t \end{cases} = \begin{cases} 0 & t \le -\frac{a}{2} \\ a t & t \ge \frac{a}{2} \end{cases}$$

(h) $y(t) = x(t) * [\text{sgn}\, t + u(-t-1)] = x(t) * [-u(t+1) + 2u(t)]$

$$= -\frac{(t+1)^2}{2} u(t+1) + t^2 u(t)$$

(i) $y(t) = [u(t+1) - u(t-1)]\,\text{sgn}\, t * u(t)$

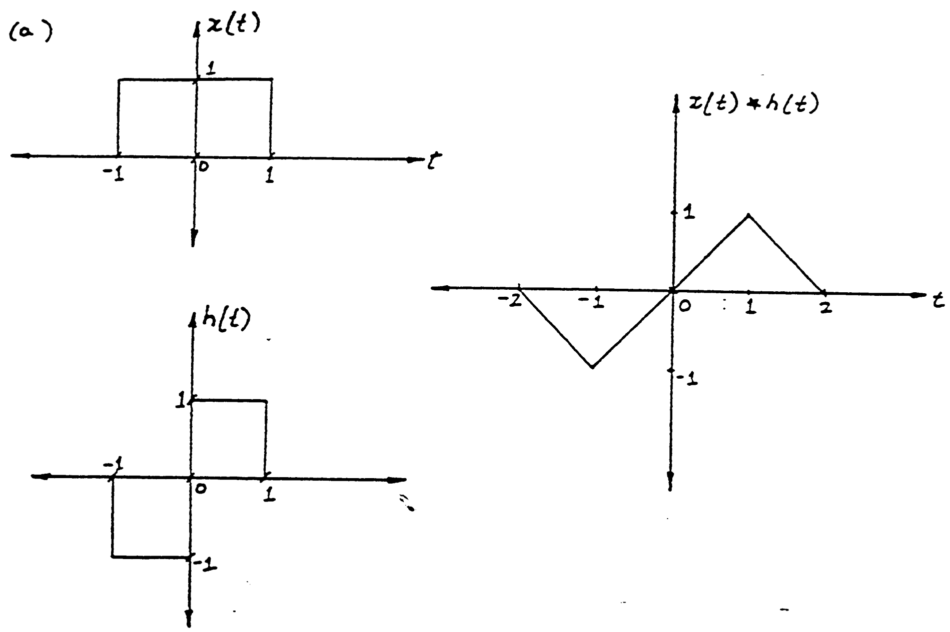
$$= [-u(t+1) + 2u(t) - u(t-1)] * u(t)$$

$$= \begin{cases} 0 & t \le -1 \\ -t-1 & -1 \le t \le 0 \\ t-1 & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases}$$

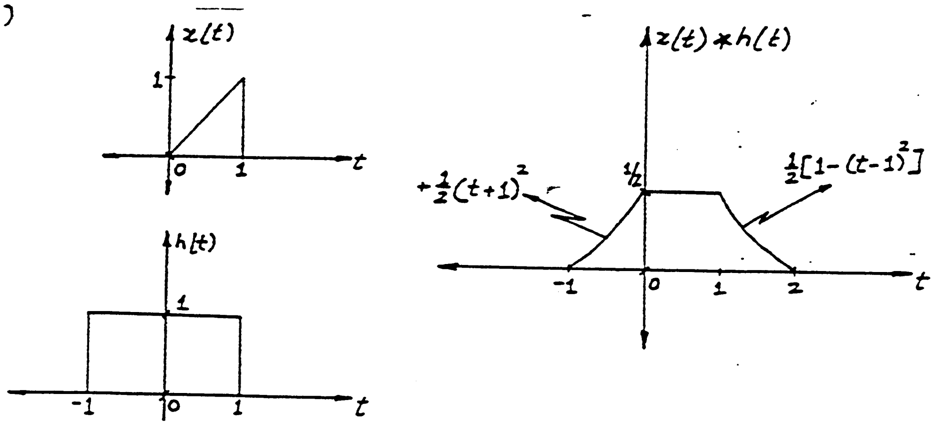
(j) $y(t) = u(t) * \delta'(t) = \int_{-\infty}^{\infty} u(t-\tau)\,\delta'(\tau)\,d\tau$

$$= \int_{-\infty}^{t} \delta'(\tau)\,d\tau = \delta(t)$$

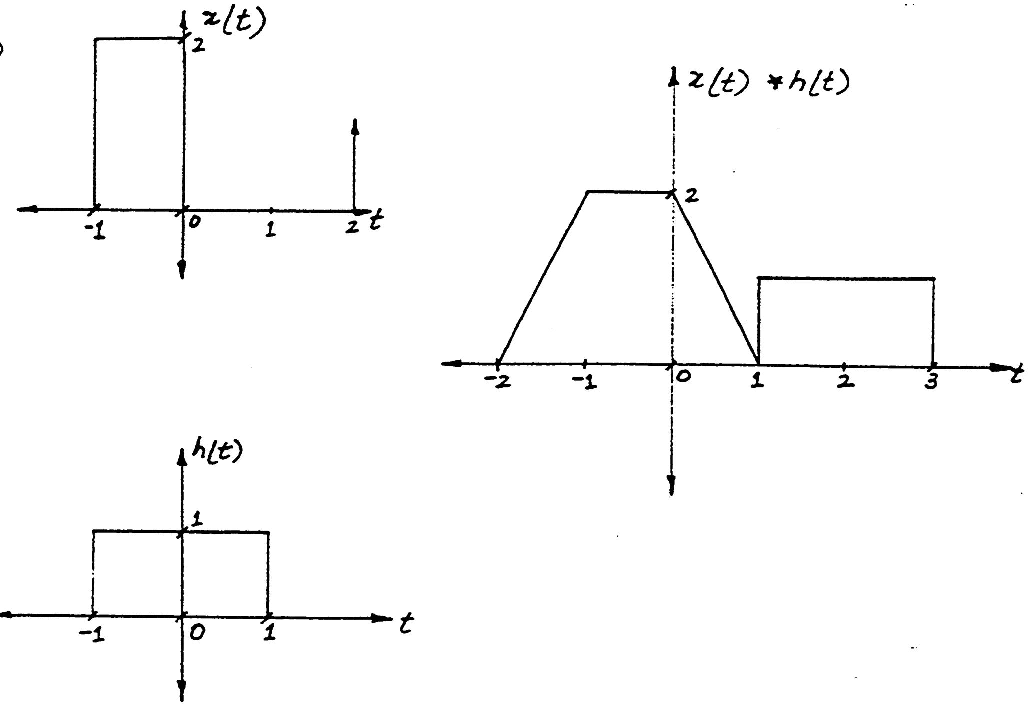
2.4 (a)



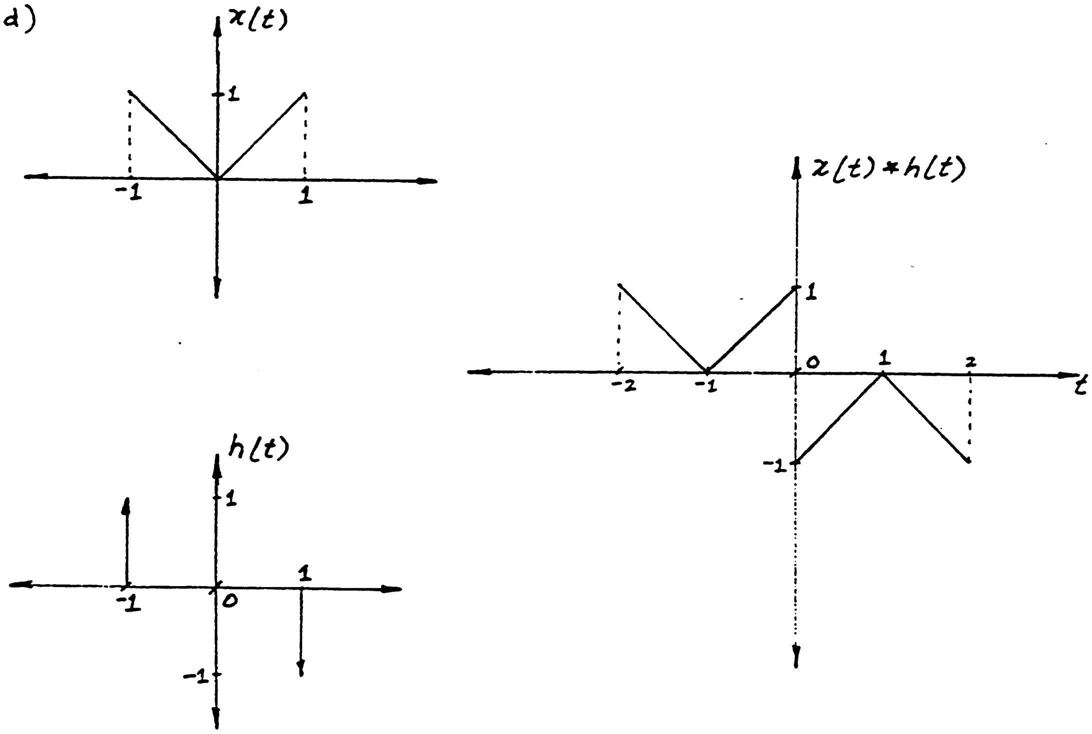
(b)



(c)


x(t)
2
-1
0
1
2
t
h(t)
1
-1
0
1
t
x(t) * h(t)
2
-2
-1
0
1
2
3
t

(d)


x(t)
1
-1
1
h(t)
1
-1
0
1
-1
x(t) * h(t)
1
0
1
2
-2
-1
-1
t

2.5 We can use the results of Example (2.3.4) to solve most of these problems.

(a) $y(t) = [e^{-t} - e^{-2t}]u(t)$

(b) $y(t) = \int_0^t \tau e^{-\tau} d\tau = [1 - (t+1)e^{-t}]u(t)$

(c) $y(t) = \int_0^t [e^{-\tau} + 1] d\tau = [1 - e^{-t} + t]u(t)$

(d) $y(t) = [e^{-2t}u(t) + \delta(t)] * u(t)$

$= e^{-2t}u(t) * u(t) + u(t)$

$= \frac{1}{2}(1 - e^{-2t})u(t) + u(t)$

(e) $y(t) = e^{-at}u(t) * [u(t) - e^{-at}u(t-b)]$

$= e^{-at}u(t) * [u(t) - e^{-ab}e^{-a(t-b)}u(t-b)]$

$= [1 - e^{-at}]u(t) - (t-b)e^{-at}u(t-b)$

(f) $y(t) = [\delta(t-1) + e^{-t}]u(t) * e^{-2t}u(t)$

$= e^{-2(t-1)}u(t-1) + [e^{-t} - e^{-2t}]u(t)$

2.6 (a) $x(t) * y(-t) = \int_{-\infty}^{\infty} x(\tau)\, y(-(t-\tau))\, d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)\, y(\tau - t)\, d\tau = R_{xy}(t)$$

(b) $R_{yx}(t) = \int_{-\infty}^{\infty} y(\tau)\, x(\tau - t)\, d\tau$

$$= \int_{-\infty}^{\infty} x(\tau')\, y(\tau' + t)\, d\tau' \neq R_{xy}(t)$$

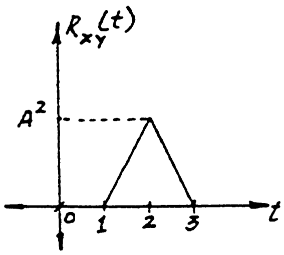
(c) $R_{yx}(-t) = \int_{-\infty}^{\infty} x(\tau')\, y(\tau' - t)\, d\tau' = R_{xy}(t)$

2.7

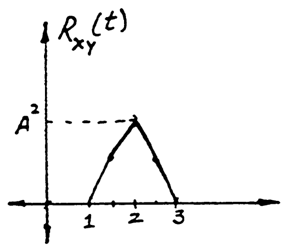
$$R_{xy}(t) = \begin{cases} 0 & t \le 1 \\ A(A+B)(t-1) & 1 < t \le \frac{3}{2} \\ \frac{A(A+B)}{2} + A(A-B)\left(t - \frac{3}{2}\right) & \frac{3}{2} < t \le 2 \\ \frac{A(A-B)}{2} + A(A+B)\left(\frac{5}{2} - t\right) & 2 < t \le \frac{5}{2} \\ A(A-B)(3-t) & \frac{5}{2} < t \le 3 \\ 0 & t > 3 \end{cases}$$

$$
R_{xy}(t) = \begin{cases}
0 & t \le 1 \\
A^2\left(1+\frac{B}{A}\right)(t-1) & 1 < t \le \frac{3}{2} \\
\frac{A^2}{2}\left(1+\frac{B}{A}\right) + A^2\left(1-\frac{B}{A}\right)\left(t-\frac{3}{2}\right) & \frac{3}{2} < t \le 2 \\
\frac{A^2}{2}\left(1-\frac{B}{A}\right) + A^2\left(1+\frac{B}{A}\right)\left(\frac{5}{2}-t\right) & 2 < t \le \frac{5}{2} \\
A^2\left(1-\frac{B}{A}\right)(3-t) & \frac{5}{2} < t \le 3 \\
0 & t > 3
\end{cases}
$$

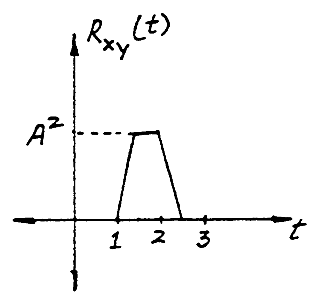
$\frac{B}{A} = 0.0$



$\frac{B}{A} = 0.1$



$\frac{B}{A} = 1$



2.8 (a) $R_x(t) = \int_{-\infty}^{\infty} x(\tau)\, x(\tau+t)\, d\tau$

$$R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt = E$$

(b) $R_x(t) \le \left(\int_{-\infty}^{\infty} x^2(\tau)\, d\tau \int_{-\infty}^{\infty} x^2(\tau+t)\, d\tau \right)^{1/2}$

$$= \left(\int_{-\infty}^{\infty} x^2(\tau)\, d\tau \int_{-\infty}^{\infty} x^2(\tau')\, d\tau' \right)^{1/2}$$

$$= R(0) \qquad \text{Q.E.D.}$$

(c) $R_z(t) = \int_{-\infty}^{\infty} z(\tau)\, z(\tau + t)\, d\tau$

$= \int_{-\infty}^{\infty} (x(\tau) + y(\tau))(x(\tau + t) + y(\tau + t))\, d\tau$

$= R_x(t) + R_{yx}(t) + R_{xy}(t) + R_y(t)$

2.9 $R_y(t) = \int_{-\infty}^{\infty} y(\tau)\, y(\tau + t)\, d\tau$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau - \mu)\, h(\mu)\, d\mu \int_{-\infty}^{\infty} x(\tau + t - \nu) h(\nu)\, d\nu\, d\tau$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\mu)\, h(\nu) \int_{-\infty}^{\infty} x(\tau - \mu)\, x(\tau + t - \nu)\, d\tau\, d\mu\, d\nu$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\mu) h(\nu)\, R_x(t - \nu + \mu)\, d\mu\, d\nu$

$= \int_{-\infty}^{\infty} h(\mu)\, G(t + \mu)\, d\mu$

where $G(\eta) = \int_{-\infty}^{\infty} h(\nu)\, R_x(\eta - \nu)\, d\nu$

$= R_x(t) * h(t)$

but the last integral is $h(-t) * G(t)$

therefore

$R_x(t) = R_x(t) * h(t) * h(-t)$

2.10 $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \exp[j\omega(t-\tau)]\, d\tau$$

$$= \exp[j\omega t] \int_{-\infty}^{\infty} h(\tau) \exp[-j\omega\tau]\, d\tau$$

$$= \exp[j\omega t]\, H(\omega)$$

2.11 (a) $h(t) = e^{-3t} \sin t\, u(t)$, Causal (C), stable (S).

(b) $h(t) = e^{4t} u(-t)$, Noncausal (NC), S

(c) $h(t) = -t\, e^{-t} u(-t)$, NC, unstable (U.S.)

(d) $h(t) = e^{-2|t|}$, NC, S

(e) $h(t) = |t-2|\, e^{-2|t|}$, NC, S

(f) $h(t) = \text{rect}\left(\frac{t}{2}\right)$, NC, S

(g) $h(t) = \delta(t) + e^{-3t} u(t)$, C, S

(h) $h(t) = \delta'(t) + e^{-2t}$, N.C., US

(i) $h(t) = \delta'(t) + e^{-2|t|}$, N.C., US

(j) $h(t) = (1-t)\, \text{rect}(t/3)$, NC, S

2.12 (a) $h(t) = \delta(t+2)$ is invertible with inverse

$h_I(t) = \delta(t-2)$

(b) $h(t) = u(t)$ is invertible with inverse

$h_I(t) = \delta'(t)$

(c) $h(t) = \delta(t-3)$ is invertible with inverse

$h_I(t) = \delta(t+3)$

(d) $h(t) = \text{rect}\,(t/4)$ is not invertible The two inputs $x_1(t) = \sin \frac{\pi}{2} t$ and $x_2 = \sin \pi t$ produce the same output $y(t) = 0$

(e) $h(t) = \exp[-t]\, u(t)$ is invertible and the inverse system is $h_I(t) = \delta'(t) + \delta(t)$

2.13 (a) No, this would contradict the assumption that system I is the optimum operation on $x(t)$.

(b) No, this would contradict the statement that the second operation in system III is optimum.

(c) These systems are identical from the performance view point.

(d) No, need only the system to be invertible.

2.14 $h(t) = h_5(t) * [h_1(t) + h_2(t) * h_3(t)] + h_4(t)$

$$= e^{-3t}u(t) * [e^{-2t}u(t) + [e^{-t} - e^{-2t}]u(t) + \delta(t)$$

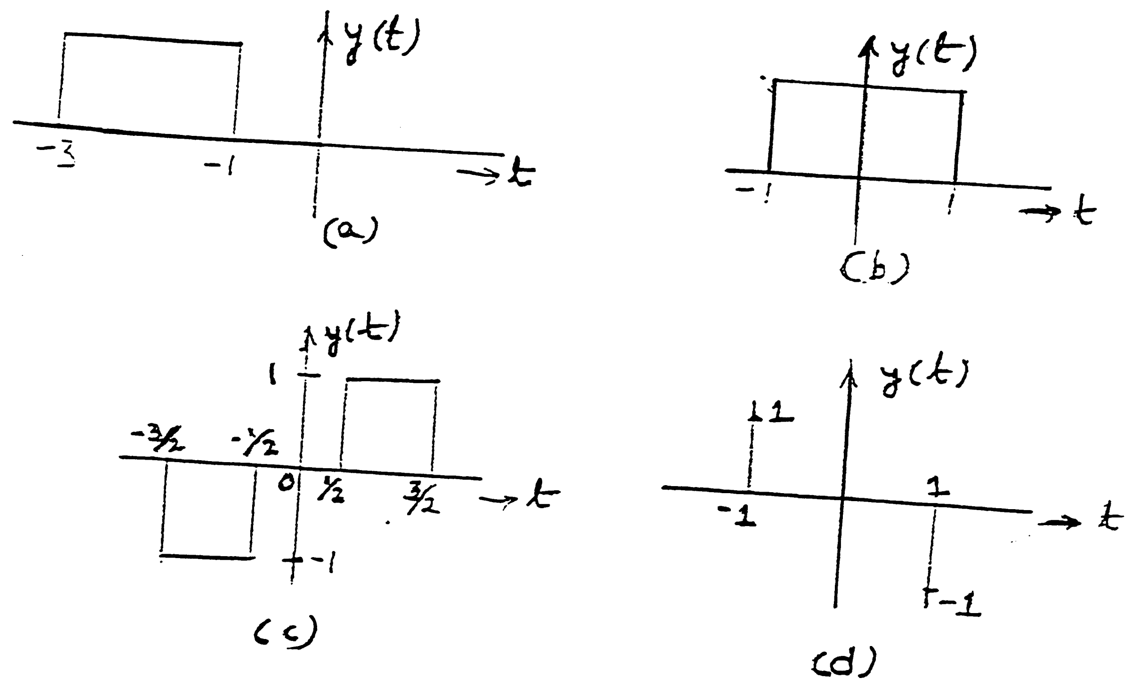
$$= e^{-3t}u(t) * [e^{-t}u(t) + \delta(t)]$$

$$= \frac{1}{2}[e^{-t} - e^{-3t}]u(t) + e^{-3t}u(t)$$

$$= \frac{1}{2}[e^{-t} + e^{-3t}]u(t)$$

Stable

2.15



2.16 $\frac{L}{R} h'(t) + h(t) = \delta(t)$

$$h(t) = \frac{R}{L} \exp[-Rt/L]\, u(t)$$

2.17 $RC \frac{dh(t)}{dt} + h(t) = \delta(t)$

(a) If $h(t) = \frac{1}{RC} \exp[-t/RC]\, u(t)$

$x(t) = u(t - \tau/2)$

(b) If $y(t) = [1-\exp[-(t-\theta/2)/RC]]u(t-\theta/2)$

$$x(t) = u(t+\tau/2)$$

$$y(t) = (1-\exp[-(t+\theta/2)/RC])u(t+\theta/2)$$

(c) If $x(t) = \text{rect}(t/\tau) = y(t+\tau/2) - u(t-\tau/2)$

$$\therefore y(t) = \text{rect}(\theta/2) - \exp[-(t+\theta/2)/RC]u(t+\theta/2)$$

$$+ \exp[-(t-\theta/2)/RC]u(t-\theta/2)$$

2.18 $$\frac{dh(t)}{dt} + \frac{h(t)}{RC} = \delta'(t)$$

$$h(t) = \delta(t) - \exp(-t/RC)u(t)$$

(a) If $x(t) = u(t-\theta/2)$

$$y(t) = u(t-\theta/2) + [RC\exp[-(t-\theta/2)/RC]-1]u(t-\theta/2)$$

$$= RC\exp[-(t-\theta/2)/RC]u(t-\theta/2)$$

(b) If $x(t) = u(t+\theta/2)$

$$y(t) = RC\exp[-(t+\theta/2)/RC]u(t+\theta/2)$$

(c) If $x(t) = \text{rect}(t/\theta) = u(t+\theta/2) - u(t-\theta/2)$

$$y(t) = RC\exp[-(t+\theta/2)/RC]u(t+\theta/2)$$

$$-RC\exp[-(t-\theta/2)/RC]u(t-\theta/2)$$

2.19 Let $x_1(t)$ and $x_2(t)$ be two input signals, and let $y_1(t)$ and $y_2(t)$ be the corresponding responses. That is

$$\frac{d^N y_1(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t)\frac{d^k y_1(t)}{dt^k} = \sum_{k=0}^{M} b_k(t)\frac{d^k x_1(t)}{dt^k} \qquad (1)$$

$$\frac{d^N y_2(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t)\frac{d^k y_2(t)}{dt^k} = \sum_{k=0}^{M} b_k(t)\frac{d^k x_2(t)}{dt^k} \qquad (2)$$

and also $y_1(t)$ and $y_2(t)$ must satisfy

$$y_1^{(k)}(t_0) = y_2^{(k)}(t_0) = 0 \qquad (3)$$

Consider next the input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, where α and β are any complex numbers. Then, using eqns (1) and (2), it is not difficult to see that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ satisfies the differential equation

$$\frac{d^N y_3(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t) \frac{d^k y_3(t)}{dt^k} = \sum_{k=0}^{M} b_k(t) \frac{d^k x_3(t)}{dt^k}$$

and also, from eq. (3)

$$y_3^{(k)}(t_0) = \alpha y_1^{(k)}(t_0) + \beta y_2^{(k)}(t_0) = 0$$

Therefore, $y_3(t)$ is the response to $x_3(t)$, and thus the system is linear.

2.20 Let $y_1(t)$ be the response to an input $x_1(t)$, which is zero for $t < t_0$. That is

$$\frac{d^N y_1(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y_1(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x_1(t)}{dt^k} \qquad (1)$$

$$y_1^{(k)}(t_0) = 0 \qquad (2)$$

Now consider the input

$$x_2(t) = x_1(t-\tau) \qquad (3)$$

note that $x_2(t)$ is zero for $t < t_0 + \tau$ Therefore, the response $y_2(t)$ to this input must satisfy the differential equation

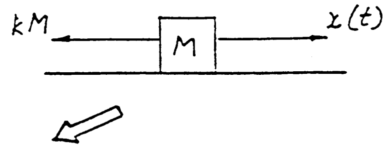
$$\frac{d^N y_2(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y_2(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x_2(t)}{dt^k} \qquad (4)$$

with the initial conditions

$$y_2^{(k)}(t_0+\tau) = 0 \qquad (5)$$

Using eqns. (1) and (2), it is straightforward to show that $y_1(t-\tau)$ satisfies eqns. (4) and (5) and thus that

$$y_2(t) = y_1(t-\tau)$$



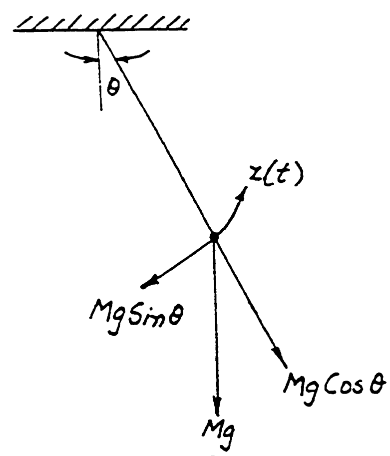
2.21 $$M\frac{d^2y(t)}{dt^2} = x(t) - kM$$

or $$M\frac{d^2y(t)}{dt^2} + kM = x(t)$$

Assuming that $y(t_0) = y'(t_0) = y''(t_0) = 0$ then according to problem 2.19, the system is linear, also according to problem 2.20, the system is time-invariant This system can be time-varying if either k or M is changing with time.

2.22 $$M\ell\,\theta''(t) = x(t) - Mg\sin(\theta(t))$$

$$\theta''(t) + \frac{g}{\ell}\sin(\theta(t)) = \frac{x(t)}{M\ell}$$



No, this system is not linear

If $\theta(t)$ is small such that $\sin(\theta(t)) \doteq \theta(t)$, then

$$\theta''(t) + \frac{g}{\ell}\theta(t) = \frac{x(t)}{M\ell}$$

which is a linear system.

2.23 With reference to the figure, let $y_1 = v_1'' + v_1'$. Then

$v_1'' = x - 2v_1' - v_1$

$y_1 = 2v_1'' + v_1' = 2[x - 2v_1' - v_1] + v_1'$

$= 2x - 3v_1' - 2v_1$ (a)

$y_1' = 2x' - 3v_1'' - 2v_1' = 2x' - 3[x - 2v_1' - v_1] - 2$

$= 2x' - 3x + 4v_1' + 3v_1$ (b)

$y_1'' = 2x'' - 3x' + 4v_1'' + 3v_1'$

$= 2x'' - 3x' + 4[x - 2v_1' - v_1] + 3v_1'$

$= 2x'' - 3x' + 4x - 5v_1' - 4v_1$ (c)

$\therefore y_1'' + 2y_1' + y_1 = 2x'' + x'$ by adding (a), 2(b), (c).

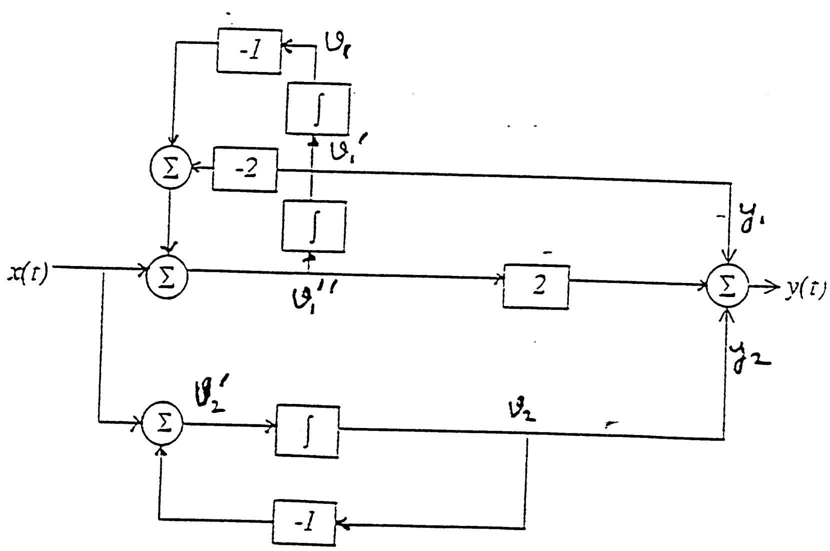
Let $y_2 = v_2$ (d), $y_2' = x - v_2 = x - y_2$ (e)

$\therefore y_2'' = x' - y_2' = x' - x + y_2$ (f)

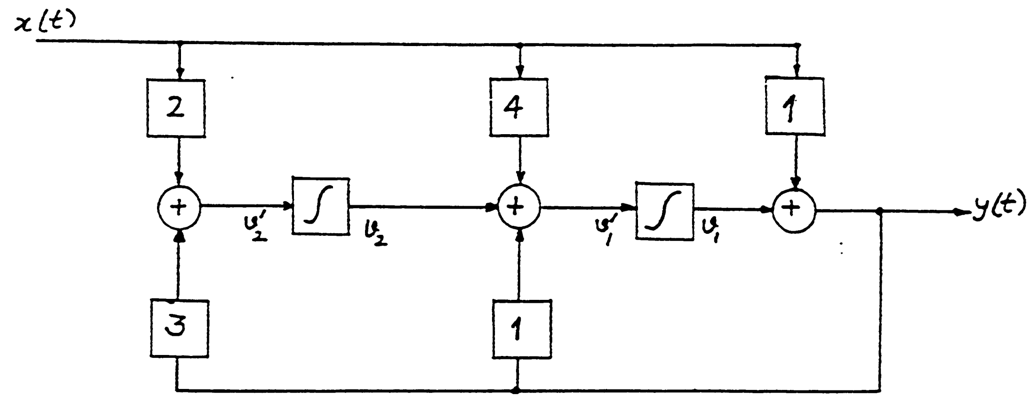
$\therefore y_2'' + 2y_2' + y_2 = x' + x$ by adding (d), 2(e), (f).

$\therefore (y_1 + y_2)'' + 2(y_1 + y_2)' + (y_1 + y_2) = 2x'' + 2x' + x$

or $y'' + 2y' + y = 2x'' + 2x' + x$



2.24



$$v_2'(t) = 2x(t) + 3y(t) \qquad (1)$$

$$v_1'(t) = 4x(t) + y(t) + v_2(t) \qquad (2)$$

$$y(t) = v_1(t) + x(t) \qquad (3)$$

From (1), (2) and (3)

$$y(t) = \int (4x(t) + y(t) + v_2(t))\,dt + x(t)$$

$$= \int \left(4x(t) + y(t) + \int (2x(t) + 3y(t))\,dt\right)dt + x(t)$$

or

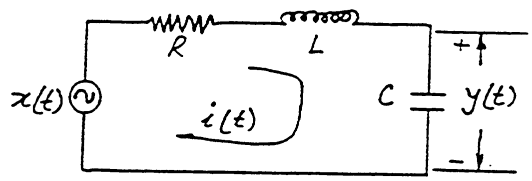
$$y'(t) = 4x(t) + y(t) + \int (2x(t) + 3y(t))\,dt + x'(t)$$

$$y''(t) = 4x'(t) + y'(t) + 2x(t) + 3y(t) + x''(t)$$

Rearranging terms, yields

$$y''(t) - y'(t) - 3y(t) = 2x(t) + 4x'(t) + x''(t)$$

2.25 (a)



$$V_L(t) = L\frac{di(t)}{dt} \qquad i = C\frac{d\,y(t)}{dt}$$

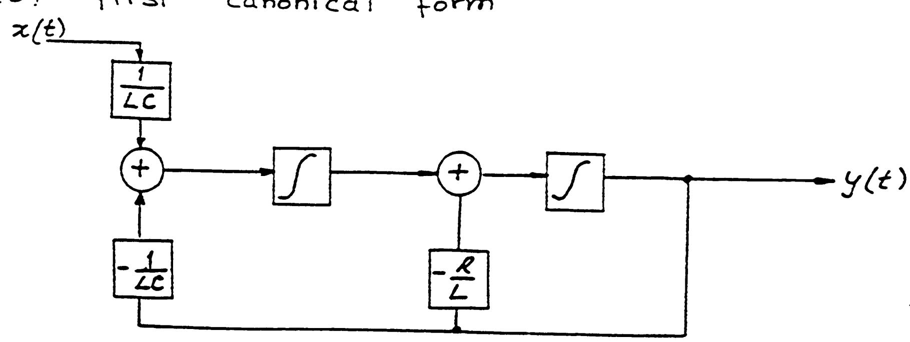
$$x(t) = R\,i(t) + L\frac{di(t)}{dt} + y(t)$$

or $$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

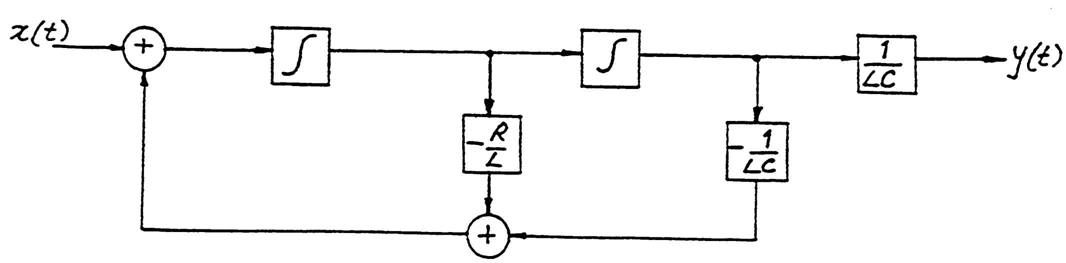
$$\Rightarrow \quad y''(t) + \frac{R}{L}y'(t) + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

$$a_0 = \frac{1}{LC}, \quad a_1 = \frac{R}{L}, \quad b_0 = \frac{1}{LC}$$

(b) first canonical form

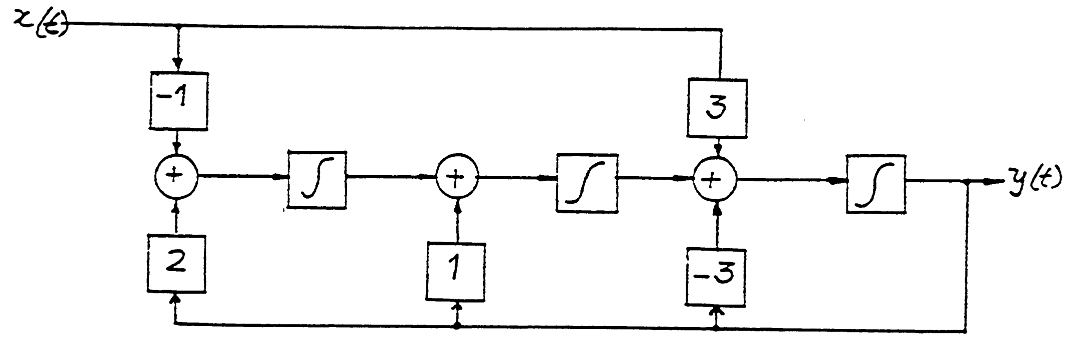


(c) second canonical form

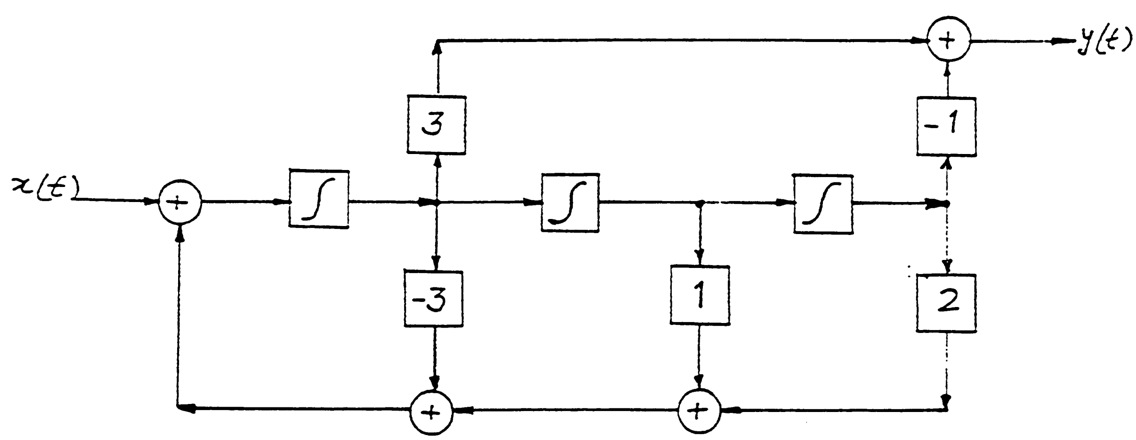


2.26 $a_2 = 3$, $a_1 = -1$, $a_0 = -2$, $b_2 = 3$, $b_1 = 0$, $b_0 = -1$

first canonical form



second canonical form



2.27 $i_c(t) = c\dfrac{dV_c(t)}{dt}$ $\qquad V_R(t) = R\,i_c(t) = y(t)$

$$x(t) = V_c(t) + y(t) = y(t) + \frac{1}{RC}\int y(t)\,dt$$

$$\Longrightarrow \quad y'(t) + \frac{1}{RC}\,y(t) = x'(t)$$

$$h'(t) + \frac{1}{RC}\,h(t) = \delta'(t) \qquad \text{let} \quad h_p(t) = D\,\delta(t)$$

$$\Longrightarrow \quad h(t) = B\,e^{-\frac{t}{RC}}\,u(t) + D\,\delta(t)$$

or $\quad B\,\delta(t) + D\,\delta'(t) + \dfrac{D}{RC}\,\delta(t) = \delta'(t) \Longrightarrow D = 1,\ B = -\dfrac{1}{RC}$

$$\Longrightarrow \quad h(t) = \delta(t) - \frac{1}{RC}\,e^{-\frac{t}{RC}}\,u(t)$$

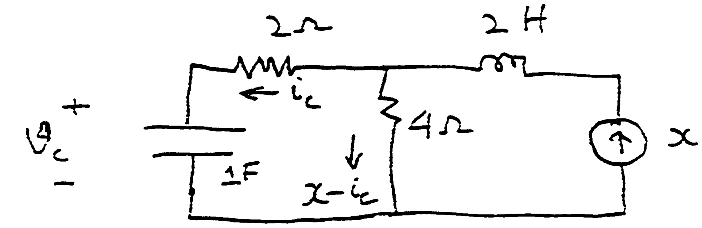
2.28 The first form is $\underline{v}'(t) = \begin{bmatrix} -\frac{5}{2} & 1 \\ -1 & 0 \end{bmatrix} \underline{v}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t)$

$$y(t) = [1 \quad 0]\, \underline{v}(t)$$

Second form is $\underline{v}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \underline{v}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$

$$y(t) = [1 \quad 1]\, \underline{v}(t)$$

2.29



Going around the first loop, we get

$$v_c - 2i_c = 4(x - i_c)$$

Also $v_c = \frac{1}{c}\int i_c\, dt \quad \therefore \; i_c = \dot{v}_c$

$$\therefore \; 6\dot{v}_c = v_c - 4x \quad \text{or} \quad \dot{v}_c = \frac{1}{6} v_c - \frac{2}{3} x$$

$i_L = x$ and is not a state variable.

2.30 First form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} x$$

$$y = [1 \quad 0 \quad 0]\, \underline{v}$$

Second form is

$$\underline{v}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \underline{x}$$

$$y = [1 \quad 1]\, \underline{v}$$

2.31 (a) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

(i) Series expansion

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -t & 0 & 0 \\ 0 & -2t & 0 \\ 0 & 0 & -3t \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} & 0 & 0 \\ 0 & 2t^2 & 0 \\ 0 & 0 & \frac{9t^2}{2} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2} + \cdots & 0 & 0 \\ 0 & 1 - 2t + 2t^2 + \cdots & 0 \\ 0 & 0 & 1 - 3t + \frac{9}{2}t^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

(ii) Caley - Hamilton

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

$$\begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} \gamma_0 & -\gamma_1 & +\gamma_2 \\ \gamma_0 & -2\gamma_1 & +4\gamma_2 \\ \gamma_0 & -3\gamma_1 & +9\gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ \frac{5}{2} & -4 & \frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} & -3e^{-2t} & +e^{-3t} \\ \frac{5}{2}e^{-t} & -4e^{-2t} & +\frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} & -e^{-2t} & +\frac{1}{2}e^{-3t} \end{bmatrix}$$

$$\exp At = \gamma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \gamma_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \gamma_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\exp At = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

(b) $$A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} -1 & 6 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & -8 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Series expansion

$$\exp At = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} t + \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{t^2}{2}$$

$$+ \begin{bmatrix} -1 & 6 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \frac{t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots, & 2t - \frac{4}{2}t^2 + \frac{6}{3!}t^3 + \cdots, & -t + \frac{2t^2}{2!} - \frac{3t^3}{3!} + \cdots \\ 0 & 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots, & 0 \\ 0 & 0 & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 2t\,(e^{-t}) & -t\,(e^{-t}) \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

(ii) Caley - Hamilton

the eigenvalues of the matrix A are $-1, -1, -1$

$$e^{At} = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$e^{-t} = \gamma_0 - \gamma_1 + \gamma_2$$

$$t e^{-t} = \gamma_1 - 2\gamma_2$$

$$t^2 e^{-t} = 2\gamma_2$$

$$\Longrightarrow \quad \gamma_2 = \frac{t^2}{2} e^{-t}$$

$$\gamma_1 = te^{-t} + t^2 e^{-t}$$

$$\gamma_0 = e^{-t} + te^{-t} + t^2 e^{-t} - \frac{t^2}{2} e^{-t} = e^{-t} + te^{-t} + \frac{t^2}{2} e^{-t}$$

$$\exp At = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$= \begin{bmatrix} e^{-t} & 2te^{-t} & -te^{-t} \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

(c) $$A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 8 \\ 0 & -2 & 5 \end{bmatrix} \qquad A^3 = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 5 & -12 \\ 0 & 3 & -7 \end{bmatrix}$$

(i) series expansion

$$\exp At = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} t + \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 8 \\ 0 & -2 & 5 \end{bmatrix} \frac{t^2}{2!}$$

$$+ \begin{bmatrix} -1 & 0 & 3 \\ 0 & 5 & -12 \\ 0 & 3 & -7 \end{bmatrix} \frac{t^3}{3!} + \cdots$$

$$
= \begin{bmatrix} 1 - t + \frac{t^2}{2} + \cdots & t - \frac{t^2}{2!} + \cdots & -t + \frac{3t^3}{3!} + \cdots \\ 0 & 1 + t - \frac{3t^2}{2!} + \frac{5t^3}{3!} + \cdots & -4t + \frac{8t^2}{2!} - \frac{12}{3!} t^3 \cdots \\ 0 & t - \frac{2t^2}{2!} + \frac{3t^3}{3!} + \cdots & 1 - 3t + \frac{5t^2}{2} - \frac{7}{3!} t^3 \cdots \end{bmatrix}
$$

$$
= \begin{bmatrix} e^{-t} & (t + \frac{t^2}{2})e^{-t} & (t + t^2)e^{-t} \\ 0 & (1+2t)e^{-t} & -4te^{-t} \\ 0 & te^{-t} & (1-2t)e^{-t} \end{bmatrix}
$$

(iii) Cayley - Hamilton

the eigenvalues of $A = -1, -1, -1$

$$e^{At} = \gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2 , \quad \text{for } \lambda_i = -1$$

$$\Longrightarrow \quad e^{-t} = \gamma_0 - \gamma_1 + \gamma_2$$

$$te^{-t} = \gamma_1 - 2\gamma_2$$

$$t^2 e^{-t} = 2\gamma_2 \quad \Longrightarrow \quad \gamma_2 = \frac{t^2}{2} e^{-t}$$

$$\gamma_1 = te^{-t} + t^2 e^{-t} = (t + t^2) e^{-t}$$

$$\gamma_0 = e^{-t} + te^{-t} + t^2 e^{-t} - \frac{t^2}{2} e^{-t} = (1 + t + \frac{t^2}{2}) e^{-t}$$

$$\exp(At) = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$
= \begin{bmatrix} e^{-t} & (t + \frac{t^2}{2})e^{-t} & (t + t^2)e^{-t} \\ 0 & (1+2t)e^{-t} & -4te^{-t} \\ 0 & te^{-t} & (1-2t)e^{-t} \end{bmatrix}
$$

2-32 Second canonical form gives $\underline{v}' = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{v}$$

Eigen values of A are $\lambda_1 = -2, \lambda_2 = -4$

Write $e^{-2t} = \gamma_0 + \gamma_1(-2)$

$$e^{-4t} = \gamma_0 + \gamma_1(-4)$$

to get $\gamma_0 = 2e^{-2t} - e^{-4t}$

$$\gamma_1 = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]$$

$$\therefore \Phi(t) = e^{At} = \gamma_0(t) I + \gamma_1(t) A$$

$$= (2e^{-2t} - e^{-4t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}(e^{-2t} - e^{-4t})\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2e^{-4t} \end{bmatrix}$$

$$h(t) = c\,\Phi\, b = -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{-4t} \qquad t \geq 0$$

2.33 $y''(t) + 7y'(t) + 12y(t) = x''(t) - 3x'(t) + 4x(t)$

$y'(0) = y(0) = 0 \qquad x(t) = \delta(t)$

the second canonical form of the system is given by

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} -8 & -10 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + x(t)$$

where $A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} -8 & -10 \end{bmatrix}$, $d = 1$

the eigenvalues of A are $-3, -4$

$$\begin{cases} e^{-3t} = \gamma_0 - 3\gamma_1 \\ e^{-4t} = \gamma_0 - 4\gamma_1 \end{cases} \Longrightarrow \quad \begin{aligned} \gamma_0 &= 4e^{-3t} - 3e^{-4t} \\ \gamma_1 &= e^{-3t} - e^{-4t} \end{aligned}$$

$$\exp At = \gamma_0 I + \gamma_1 A = \begin{bmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{bmatrix}$$

$$h(t) = c\exp(At)b + d\delta(t)$$

$$= \begin{bmatrix} -8 & -10 \end{bmatrix} \begin{bmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \delta(t) \qquad t \geq 0$$

$$= 24e^{-3t} - 32e^{-4t} + \delta(t) \qquad t \geq 0$$

2.34 $v'(t) = Av(t) + bx(t)$

$y(t) = cv(t) + dx(t)$

assume $z(t) = pv(t)$ and p^{-1} exists then

$v(t) = p^{-1}z(t) \qquad v'(t) = p^{-1}z'(t)$ substitute into state equations

$$\Longrightarrow \quad p^{-1}z'(t) = Ap^{-1}z(t) + bx(t)$$

$$y(t) = cp^{-1}z(t) + dx(t)$$

or

$$z'(t) = PAP^{-1}z(t) + Pbx(t) = A_1 z(t) + b_1 x(t)$$

$$y(t) = cp^{-1}z(t) + dx(t) = c_1 z(t) + d_1 x(t)$$

Q.E.D

2.35

(a) $\underline{v}' = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\underline{v} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\underline{v}$$

(b)
$$\underline{z}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$$
$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} \underline{z}$$

(c) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$PA_1 = A_1 P$ gives $\quad p_{11} = p_{22}$
$$3p_{11} + 2p_{12} + p_{21} = 0$$

$\underline{b}_1 = P\underline{b}$ gives $\quad p_{11} - p_{12} = 0$
$$p_{21} - p_{22} = 1$$

(d)
$$P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \therefore P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\hat{A} = PA_1P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{b} = Pb_1 = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{c} = c_1 P^{-1} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \end{bmatrix}$$

The state equations $\quad \underline{\hat{v}}' = \hat{A}\,\underline{\hat{v}} + \underline{\hat{b}}\,x$
$$\hat{y} = \hat{c}$$
are in diagonal form.

Chapter 3

3.1 $X_1(t) = x_{11} f_1(t) + x_{12} f_2(t)$

$$x_{11} = \int x_1(t) f_1(t)\,dt = \int_0^1 (2t)\left(\sqrt{\tfrac{3}{2}}\,t\right)dt$$

$$= 2\sqrt{\tfrac{3}{2}}\ \frac{t^3}{3}\Big|_0^1$$

$$= \frac{2}{3}\sqrt{\frac{3}{2}} = \sqrt{\frac{2}{3}}$$

$$x_{12} = \int x_1(t) f_2(t)\,dt = \sqrt{\frac{2}{3}}$$

$$\mathbf{x}_1 = \left(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$$

$$X_2(t) = x_{21} f_1(t) + x_{22} f_2(t)$$

$$x_{21} = \int x_2(t) f_1(t)\,dt = \int_1^2 2(t-2)\sqrt{\tfrac{3}{2}}\,(t-2)\,dt$$

$$= 2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2\,dt$$

$$= \sqrt{\frac{2}{3}}$$

$$x_{22} = \int x_2(t) f_2(t)\,dt = \int_1^2 2(t-2)\sqrt{\tfrac{3}{2}}\,(2-t)\,dt$$

$$= -2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2\,dt$$

$$= -\sqrt{\frac{2}{3}}$$

$$\therefore\ x_2 = \left(\sqrt{\frac{2}{3}}, -2\sqrt{\frac{2}{3}}\right)$$

$$x_3(t) = x_{31} f_1(t) + x_{32} f_2(t)$$

$$x_{31} = \int x_3(t) f_1(t)\,dt = \int_0^1 3t \sqrt{\frac{3}{2}}\, t\,dt + \int_1^2 -2(t-2)\sqrt{\frac{3}{2}}(t-2)\,dt$$

$$= 3\sqrt{\frac{3}{2}} \int_0^1 t^2 dt - 2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt$$

$$= 3\sqrt{\frac{3}{2}} \left(\frac{1}{3}\right) - 2\sqrt{\frac{3}{2}} \left.\frac{(t-2)^3}{3}\right|_1^2$$

$$= \sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}} = \sqrt{\frac{1}{6}}$$

$$x_{32} = \int x_3(t) f_2(t)\,dt = \int_0^1 3t \sqrt{\frac{3}{2}}\, t\,dt + \int_1^2 -2(t-2)\cdot\left(-\sqrt{\frac{3}{2}}(t-2)\right)dt$$

$$= 3\sqrt{\frac{3}{2}} \int_0^1 t^2 dt + 2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt$$

$$= 3\sqrt{\frac{3}{2}} \left(\frac{1}{3}\right) + 2\sqrt{\frac{3}{2}} \left.\frac{(t-2)^3}{3}\right|_1^2 = \sqrt{\frac{9}{2}} + \sqrt{\frac{2}{3}} = \frac{5}{\sqrt{6}}$$

$$\therefore\ x_3 = \left(\frac{1}{\sqrt{6}}, \frac{5}{\sqrt{6}}\right)$$

3.2 (a) Choose a_1 such that $< \psi_1, \psi_2 > = 0$. Thus

$$< x_1, x_2 + a_1 x_1 > = 0 \quad \text{or} \quad a_1 = -\frac{< x_1, x_2 >}{E_1}$$

Choose b_1, b_2 such that $< \psi_1, \psi_2 > = < \psi_2, \psi_3 > = 0$. Thus

$$< x_1, x_3 + b_1 x_1 + b_2(x_2 + a_1 x_1) > = < x_1, x_3 > + b_1 E_1$$

so that

$$b_1 = -\frac{< x_1, x_3 >}{E_1}$$

Also

$$< x_2 + a_1 x_1, x_3 + b_1 x_1 + b_2(x_2 + a_1 x_1) > = < x_2, x_3 > + a_1 < x_1, x_3 >$$
$$+ b_2[E_2 - \frac{< x_1, x_2 >^2}{E_1}]$$

so that

$$b_2 = -\frac{<x_1, x_2><x_1, x_3> - E_1<x_2, x_3>}{<x_1, x_2>^2 - E_1 E_2}$$

(b) $\psi_1 = x_1,\ E_1 = <\psi_1, \psi_1> = \frac{8}{3}$

$<x_1, x_2> = 0$, so that $\psi_2 = x_2$, and $<\psi_2, \psi_2> = E_2 = 2$

$$<x_1, x_3> = \sqrt{\frac{3}{2}}\left[\int_0^1 (1-t)t\,dt + \int_1^2 (-t^2 + 3t - 2)\right]dt = \frac{1}{\sqrt{6}}$$

$<x_2, x_3> = 0$, so that $b_1 = -\frac{1}{\sqrt{2}\sqrt{3}} \times \frac{3}{8} = -\frac{\sqrt{3}}{8\sqrt{2}}$, $b_2 = 0$

Therefore

$$\psi_3 = x_3 - \frac{\sqrt{3}}{8\sqrt{2}} x_1 = \begin{cases} \sqrt{\frac{3}{2}}\left(\frac{9}{8}t - \frac{1}{8}\right) & 0 \le t \le 1 \\ \sqrt{\frac{3}{2}}\left(\frac{9}{8}t - \frac{17}{8}\right) & 1 \le t \le 2 \end{cases}$$

and

$$<\psi_3, \psi_3> = \frac{3}{2} \times \frac{1}{64}\left[\int_0^1 (9t-1)^2 dt + \int_1^2 (9t-17)^2 dt\right] = \frac{57}{64}$$

We therefore have

$$\phi_1(t) = \sqrt{\frac{3}{8}}\,\psi_1 = \sqrt{\frac{3}{8}}\,(1-t) \quad 0 \le t \le 2$$

$$\phi_2(t) = \frac{1}{\sqrt{2}}\psi_2 = \frac{1}{\sqrt{2}} \quad 0 \le t \le 2$$

$$\phi_3(t) = \sqrt{\frac{8}{57}}\,\psi_3 = \begin{cases} \frac{8}{\sqrt{38}}\left(\frac{9}{8}t - \frac{1}{8}\right) & 0 \le t \le 1 \\ \frac{8}{\sqrt{38}}\left(\frac{9}{8}t - \frac{17}{8}\right) & 1 \le t \le 2 \end{cases}$$

3.3 (a) $\psi_1 = e^{-t}u(t),\ E_1 = \int_0^\infty e^{-2t}dt = \frac{1}{2},\ E_2 = \int_0^\infty e^{-4t}dt = \frac{1}{4}$

$$< x_1, x_2 > = \int_0^\infty e^{-3t}dt = \frac{1}{3}, \text{ so that } a_1 = -\frac{1/3}{1/2} = -\frac{2}{3}$$

and

$$\psi_2 = [e^{-2t} - \frac{2}{3}e^{-t}]u(t)$$

Now

$$< x_1, x_3 > = \int_0^\infty e^{-4t}dt = \frac{1}{4}, < x_2, x_3 > = \int_0^\infty e^{-5t}dt = \frac{1}{5}$$

so that

$$b_1 = -\frac{1/4}{1/2} = -\frac{1}{2},\quad b_2 = -\frac{\frac{1}{3}\text{x}\frac{1}{4} - \frac{1}{2}\text{x}\frac{1}{5}}{\frac{1}{9} - \frac{1}{2}\text{x}\frac{1}{4}} = -\frac{6}{5}$$

and

$$\psi_3 = [e^{-3t} - \frac{1}{2}e^{-t} - \frac{6}{5}(e^{-2t} - \frac{2}{3}e^{-t})]u(t)$$
$$= [e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}]u(t)$$

(b) $< \psi_1, \psi_1 > = \frac{1}{2},\ \phi_1 = \sqrt{2}\psi_1 = \sqrt{2}e^{-t}u(t)$

$$< \psi_2, \psi_2 > = \frac{1}{36},\ \phi_2 = 6\psi_2 = 6[e^{-2t} - \frac{2}{3}e^{-t}]u(t)$$

$$< \psi_3, \psi_3 > = \frac{1}{600},\ \phi_3 = 10\sqrt{6}\psi_2 = 10\sqrt{6}[e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}]u(t)$$

(c) Now $c_i = \int_0^\infty x\phi_i dt$

so that

$$c_1 = \frac{\sqrt{2}}{5},\ c_2 = 6[\frac{1}{6} - \frac{2}{3}\text{x}\frac{1}{5}] = \frac{1}{5},\ c_3 = 10\sqrt{6}[\frac{1}{7} - \frac{6}{5}\text{x}\frac{1}{6} + \frac{3}{10}\text{x}\frac{1}{5}] = \frac{\sqrt{6}}{35}$$

$$\hat{x}(t) = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$$

$$= [\frac{4}{35}e^{-t} - \frac{6}{7}e^{-2t} + \frac{12}{7}e^{-3t}]u(t)$$

and

E_x = Energy in $x(t) = 0.125$

$E_{\hat{x}}$ = Energy in $\hat{x}(t) = 0.1249$

E_e = Energy in $x(t) - \hat{x}(t) = 0.000102$

3.4 (a) $E_e(M) = \int_a^b \left| x(t) - \sum_{i=1}^{M} c_i \phi_i(t) \right|^2 dt$

$$\frac{\partial E_e}{\partial c_j} = \int_a^b \frac{\partial}{\partial c_j} \left| x(t) - \sum_{i=1}^{M} c_i \phi_i(t) \right|^2 dt$$

$$= \int_a^b 2\left(x(t) - \sum_{i=1}^{M} c_i \phi_i(t)\right) \frac{\partial}{\partial c_j}\left(x(t) - \sum_{i=1}^{M} c_i \phi_i(t)\right) dt$$

$$= \int_a^b 2\left(x(t) - \sum_{i=1}^{M} c_i \phi_i(t)\right)\left(-\phi_j(t)\right) dt$$

$$= -2\int_a^b x(t)\phi_j(t)\,dt + \int_a^b \sum_{i=1}^{M} c_i \phi_i(t)\phi_j(t)\,dt$$

in order to minimize the energy error, the necessary condition $\frac{\partial E_e}{\partial c_j} = 0$

$$\Rightarrow c_j = \int_a^b x(t)\phi_j(t)\,dt.$$

(b) Consider complex functions

$$E_e(M) = \int_a^b \left| X(t) - \sum_{i=1}^{M} c_i \phi_i(t) \right|^2 dt$$

$$= \int_a^b \left(X(t) - \sum_{i=1}^{M} c_i \phi_i(t) \right) \left(X(t) - \sum_{i=1}^{M} c_i \phi_i(t) \right)^* dt$$

$$\frac{\partial E_e(M)}{\partial c_j} = \int_a^b \left(X^*(t) - \sum_{i=1}^{M} c_i^* \phi_i^*(t) \right) \left(-\phi_j(t) \right) dt$$

Note that $\frac{\partial}{\partial c_j} c_j^* = 0$

Equating to zero and solving yields

$$c_j^* = \int_a^b x^*(t)\, \phi_j(t)\, dt$$

or

$$c_i = \int_a^b x(t)\, \phi_i^*(t)\, dt$$

3.5 $$\int_0^T \phi_i^*(t)\, x(t)\, dt = \int_0^T \phi_i^*(t) \sum_{k=-\infty}^{\infty} c_k \phi_k(t)\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_0^T c_k \phi_i^*(t)\, \phi_k(t)\, dt$$

$$\int x(t)\, \phi_i^*(t)\, dt = c_i E_i$$

or $$c_i = \frac{1}{E_i} \int_0^T x(t)\, \phi_i^*(t)\, dt$$

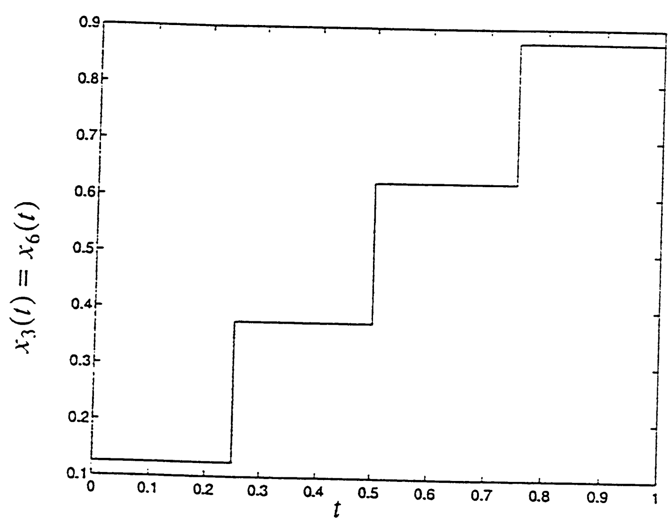
3.6 (a) We can verify that the given Walsh functions are orthonormal by directly evaluating

$$\int_0^1 wal_w(k,t)wal_w(j,t)dt, \ \ 0 \le k, j \le 6$$

(b) Use $c_k = \int_0^1 wal_w(k,t)\ t\ dt$ to get

$$c_0 = \frac{1}{2},\ c_1 = -\frac{1}{4},\ c_3 = -\frac{1}{8},\ c_2 = c_4 = c_5 = c_6 = 0$$

Thus $x_3(t) = x_6(t)$ and is as shown below.



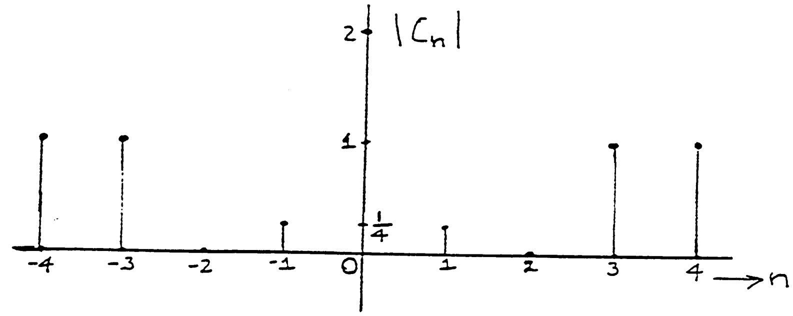
3.7 (a) With $\omega_0 = 1$, we can write

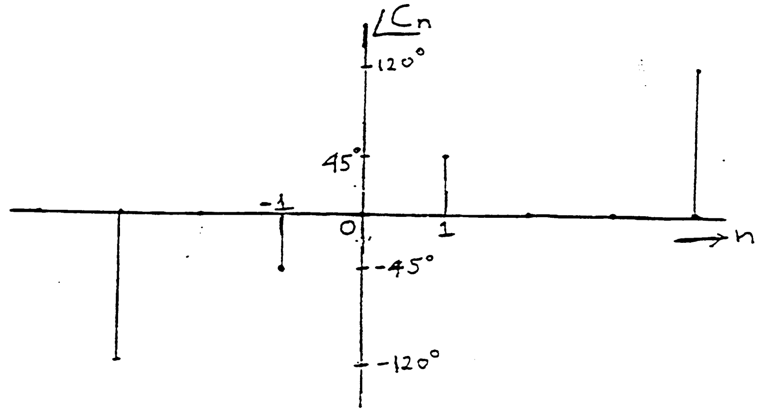
$$x(t) = 2 + \frac{1}{4}\left[e^{j(\omega_0 t + 45^o)} + e^{-j(\omega_0 t + 45^o)}\right] + \left[e^{j3\omega_0 t} + e^{-j3\omega_0 t}\right] + j\left[e^{j(4\omega_0 t + 30^o)} - e^{-j(4\omega_0 t + 30^o)}\right]$$

so that

$$c_0 = 2,\ c_1 = c_{-1}{}^* = \frac{1}{4}e^{j4^o}, c_3 = c_{-3}{}^* = 1,\ c_4 = c_{-4}{}^* = je^{j30^o} = e^{j120^o}$$

(b)





3.8

$$x(t) = \begin{cases} 0 & -\pi < t < -\frac{\pi}{2} \\ \cos t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \end{cases}$$

$$C_n = \frac{1}{T} \int_T x(t) \exp[-jnt]\,dt$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos t \exp[-jnt]\,dt$$

$$= \frac{1}{\pi(1-n^2)} \cos\frac{n\pi}{2} \qquad |n| \neq 1$$

for $n = \pm 1$ use L'Hospital's rule

$$c_1 = c_{-1} = \frac{1}{4}$$

3.9 $x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin nt\right]$

with

$$a_0 = C_0 = \frac{1}{\pi}$$

$$a_n = 2\,\mathrm{Re}\{C_n\} = \frac{2}{\pi(1-n^2)} \cos\frac{n\pi}{2} \qquad |n| \neq 1$$

$$b_n = 2\,\mathrm{Im}\{C_n\} = 0$$

3.10 $x(t) = \sin t \qquad 0 \le t \le \pi \qquad x(t+\pi) = x(t)$

$$C_n = \frac{1}{T}\int_T x(t)\exp[-j2nt]\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin t \, \exp[-j2nt]\,dt = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\exp[jt]-\exp[-jt]}{2j}\right)\exp[-j2nt]\,dt$$

$$= \frac{2}{\pi}\left(\frac{1}{1-4n^2}\right)$$

3.11 $x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos 2nt + b_n \sin 2nt\right)$

$$a_0 = C_0 = \frac{2}{\pi}$$

$$a_n = 2\,\mathrm{Re}\{C_n\} = \frac{4}{\pi}\left(\frac{1}{1-4n^2}\right)$$

$$b_n = -2\,\mathrm{Im}\{C_n\} = 0$$

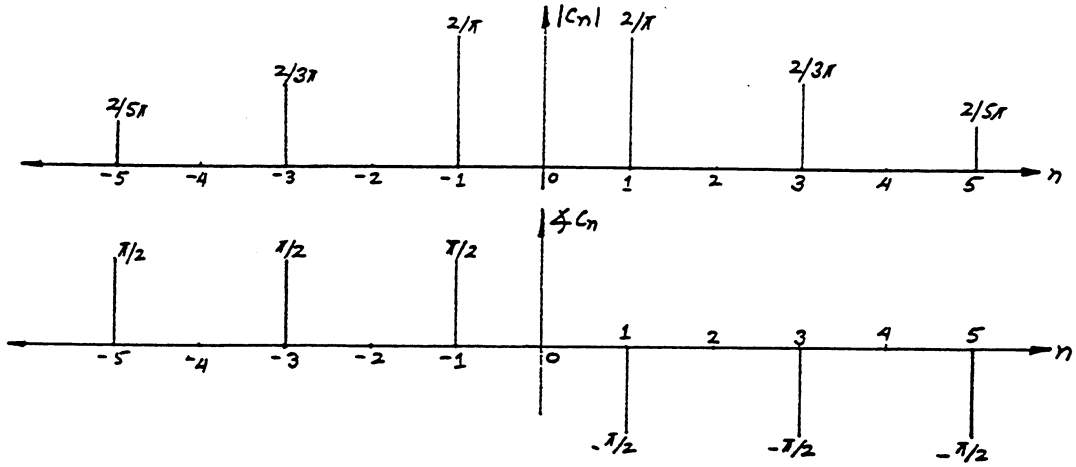
3.12(a) With $k=1$ in example 3.2.1

$C_0 = 0 \qquad C_n = \frac{2}{jn\pi} \qquad n$ odd

$= 0 \qquad n$ even

$|C_n| = \frac{2}{|n|\pi} \qquad n$ odd

$$\measuredangle C_n = \begin{cases} -\pi/2 & n = 2m-1 \quad m = 1, 2 \cdots \\ 0 & n = 2m \quad m = 0, 1, 2 \cdots \\ \pi/2 & n = -(2m-1) \quad m = 1, 2 \cdots \end{cases}$$



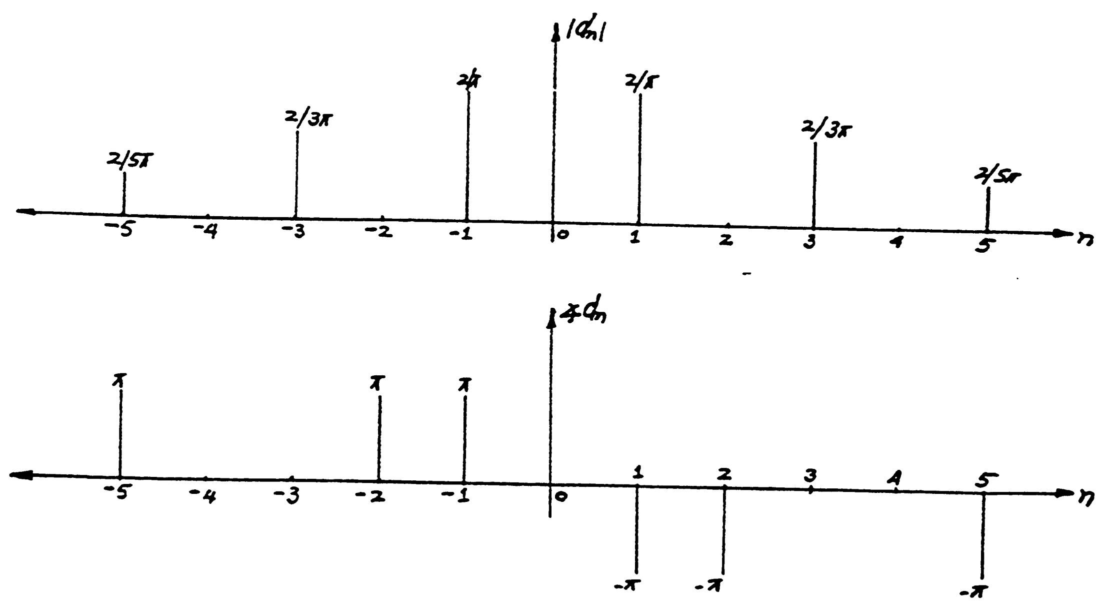
(b) The $x(t)$ in (b) is the time shift version of $x(t)$ in (a) with shift $\tau = -\frac{1}{2}$

$$d_n = c_n \exp[jn\omega_0 \tau]$$

$$= c_n \exp\left[-jn\frac{\pi}{2}\right]$$

$$\Rightarrow \quad |d_n| = |c_n|$$

$$\angle d_n = \angle c_n - \frac{n\pi}{2} \qquad (-\pi < \angle d_n < \pi)$$

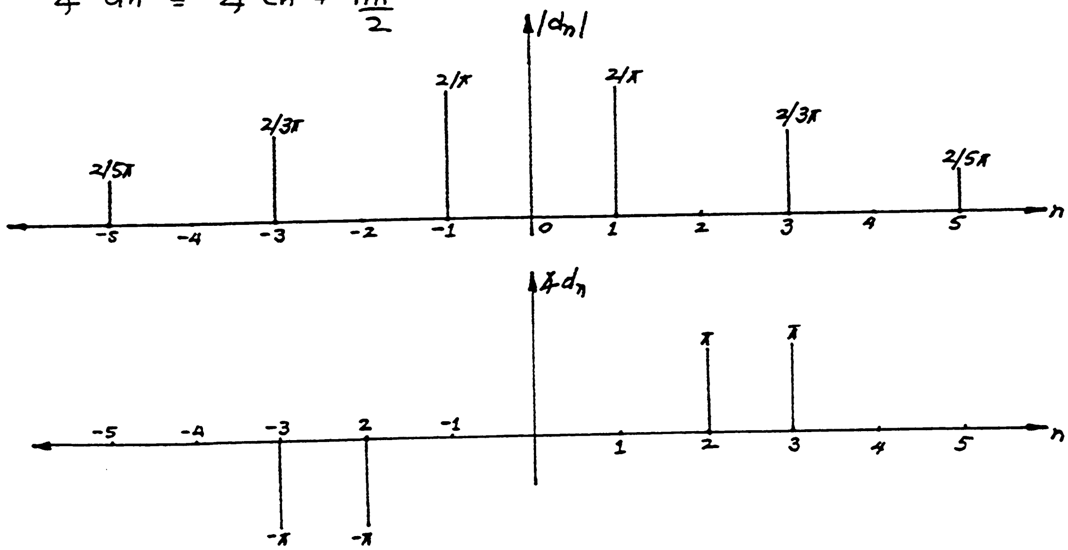


(c) The $x(t)$ in (c) is the time-shift version of $x(t)$ in (a) with shift $\tau = \frac{1}{2}$

$$d_n = c_n \exp\left[j \frac{n\pi}{2}\right]$$

$$\Rightarrow |d_n| = |c_n|$$

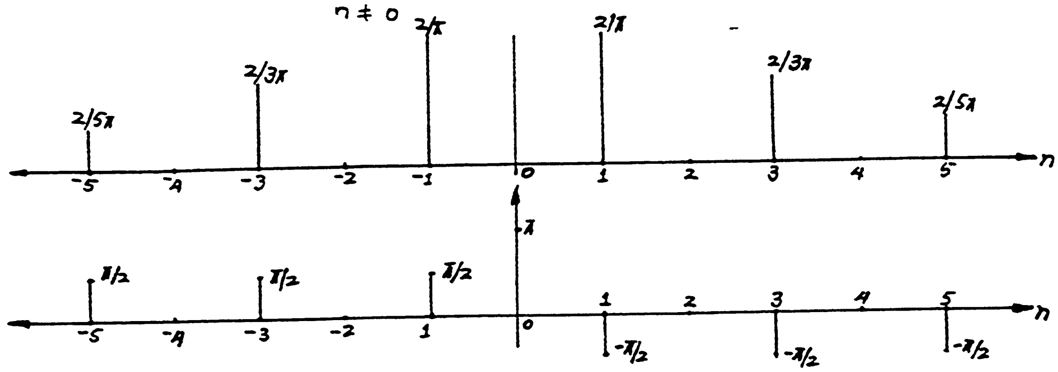
$$\measuredangle d_n = \measuredangle c_n + \frac{n\pi}{2}$$



(d) $x(t)$ in (d) is equal to $x(t)$ in (a) with a dc component -1

ie $$x(t) = -1 + \sum_{n=-\infty}^{\infty} c_n \exp[jn\pi t]$$

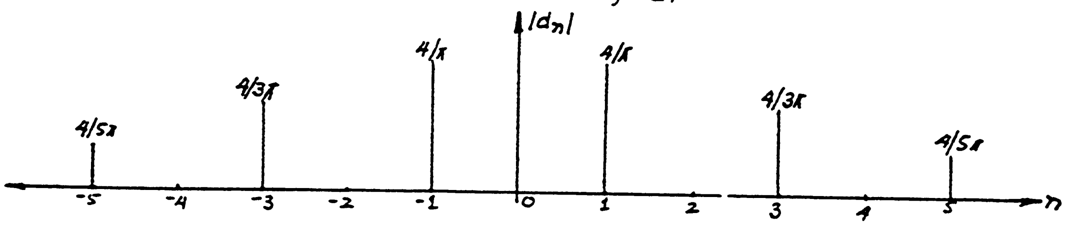
$$= -1 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n \exp[jn\pi t]$$

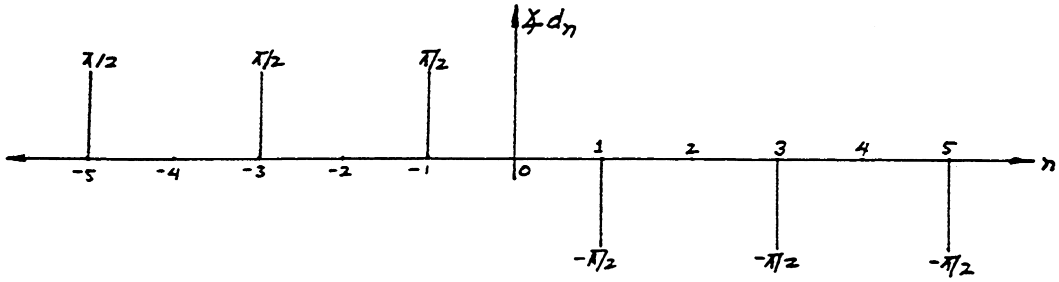


(e) $x(t)$ in (e) is equal to 2 times of $x(t)$ in (a)

$$x(t) = 2\sum_{n=-\infty}^{\infty} C_n \exp[jn\pi t]$$

$\Longrightarrow$ $d_n = 2C_n$ and $\measuredangle d_n = \measuredangle C_n$

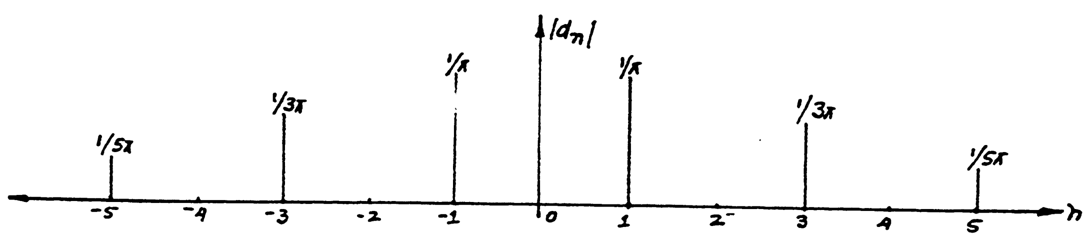


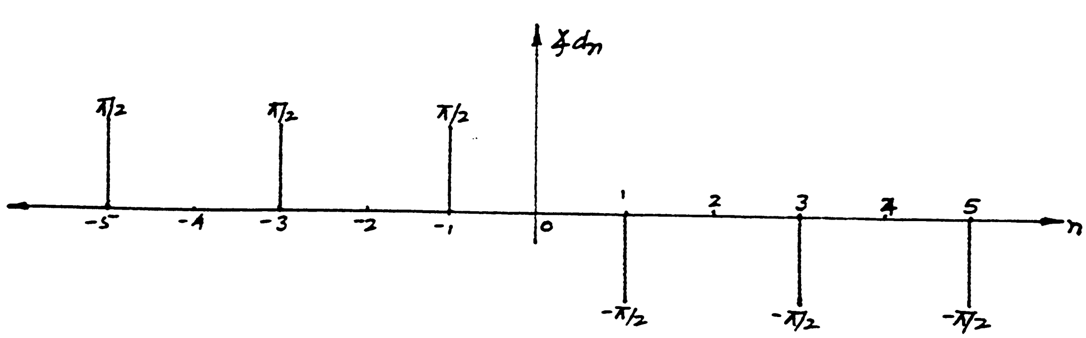


(f) $x(t)$ in (f) is equal to one half of $x(t)$ in (a)

that is, $x(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} C_n \exp[jn\pi t]$

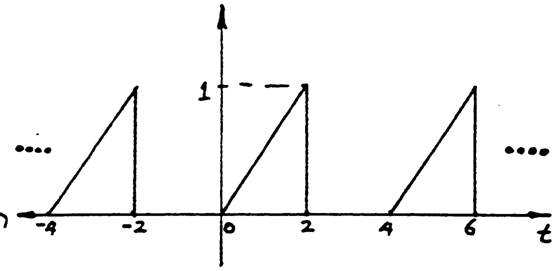
$\Longrightarrow$ $d_n = \frac{1}{2}C_n$, $\measuredangle d_n = \measuredangle C_n$



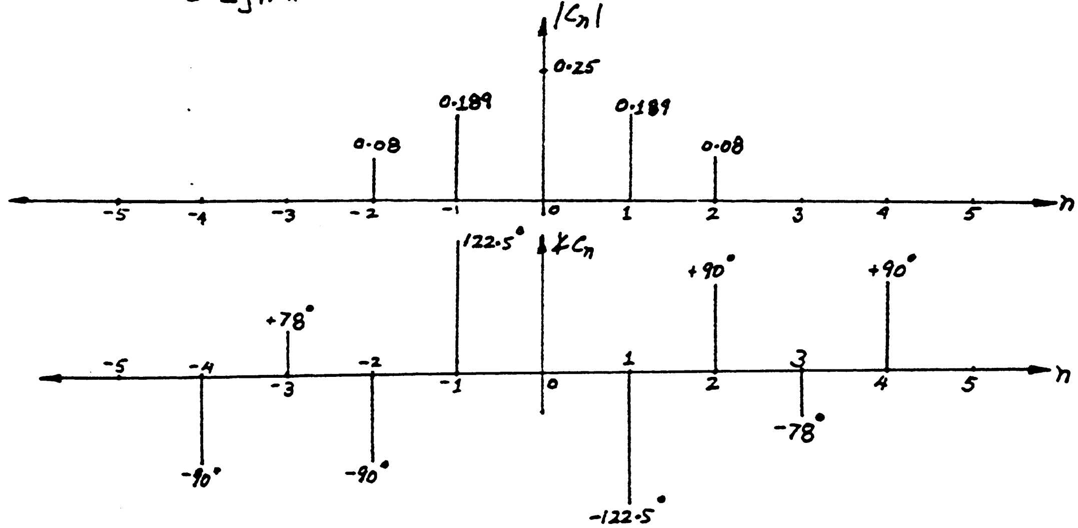


(g) $C_0 = \frac{1}{4}$

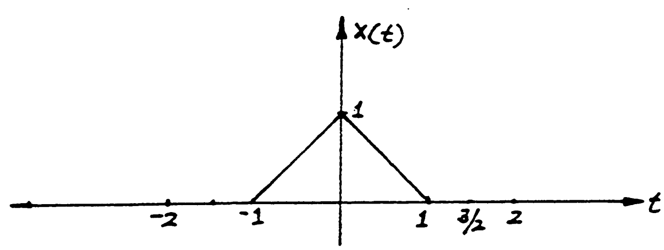
$$c_n = \frac{1}{4}\int_0^2 \frac{t}{2}\exp[-jn\frac{\pi}{2}t]\,dt$$



$$= \begin{cases} \frac{j}{2n\pi} & n \text{ even} \\ \frac{1}{2jn\pi} - \frac{1}{n^2\pi^2} & n \text{ odd} \end{cases}$$



(h)



$x(t)$ is an even function

$$C_n = \frac{1}{T}\int_T x(t)\exp[-jn\omega_0 t]\,dt$$

$$= \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t\,dt$$

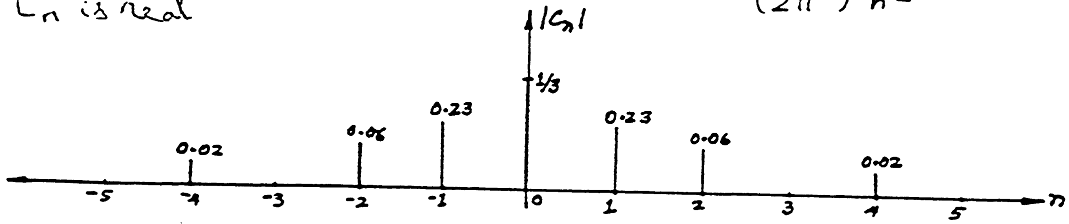
with $T = 3$ $\quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$, $\quad C_0 = 0$

$$C_n = \frac{2}{3}\int_0^1 (1-t)\cos\frac{2n\pi}{3}t\,dt$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}\sin\left(\frac{2n\pi}{3}\right) - \frac{3}{2n\pi}\left[\sin\frac{2n\pi}{3} + \frac{3}{2n\pi}\left(\cos\frac{2n\pi}{3} - 1\right)\right]\right]$$

$$= -\frac{2}{3}\left[\left(\frac{3}{2n\pi}\right)^2\left(\cos\frac{2n\pi}{3} - 1\right)\right] = -\frac{3\left[\cos\left(\frac{2n\pi}{3}\right) - 1\right]}{(2\pi^2)\,n^2}$$

C_n is real



(i) $x(t)$ is an odd symmetric function

$$C_n = \frac{1}{T}\int_T x(t)\exp[-jn\omega_0 t]\,dt$$

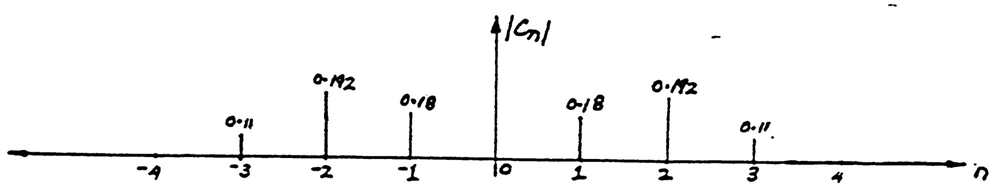
$$= -\frac{2j}{T}\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 t.dt$$

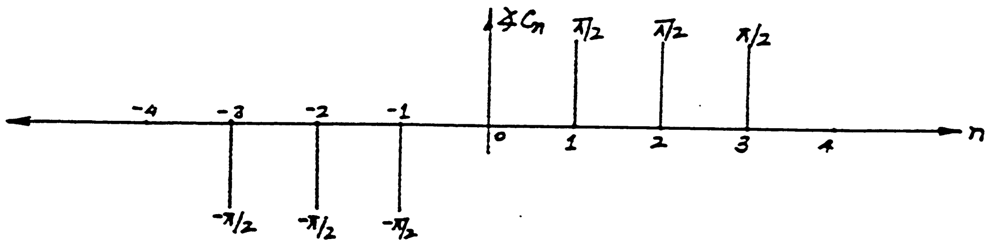
with $\quad T = 3 \quad \omega_0 = \frac{2\pi}{3}, \quad C_0 = 0$

$$C_n = -\frac{2j}{3}\int_0^1 (t-1)\sin\frac{2n\pi}{3}t\,dt$$

$$= -\frac{2j}{3}\left[\frac{3}{2n\pi}\left(\cos\frac{2n\pi}{3} - 1\right) - \frac{3}{2n\pi}\cos\frac{2n\pi}{3} + \left(\frac{3}{2n\pi}\right)^2\sin\frac{2n\pi}{3}\right]$$

$$= -\frac{2j}{3}\left[\frac{-3}{2n\pi} + \left(\frac{3}{2n\pi}\right)^2\sin\frac{2n\pi}{3}\right] = j\left[\frac{1}{n\pi} - \frac{9}{4n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$





3.13 (a) $a_n = 2\,\mathrm{Re}\{C_n\} = 0$

$$b_n = -2\,\mathrm{Im}\{C_n\} = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(b) $d_n = C_n \exp[-j\frac{n\pi}{2}]$

$$\therefore\ a_n = 2\,\mathrm{Re}\{d_n\} = \begin{cases} \frac{-4}{n\pi}\sin\frac{n\pi}{2} & n \text{ odd} \\ 0 & \end{cases}$$

$$b_n = -2\,\mathrm{Im}\{d_n\} = \begin{cases} \frac{-4}{n\pi}\cos\frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(c) $d_n = C_n \exp[jn\frac{\pi}{2}]$

$$\therefore\ a_n = 2\,\mathrm{Re}\{d_n\} = \begin{cases} \frac{4}{n\pi}\sin\frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = -2\,\mathrm{Im}\{d_n\} = \begin{cases} \frac{-4}{n\pi}\cos\frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(d) $a_0 = -1$, a_n and b_n are the same as in (a)

(e) $d_n = 2C_n$

$$\Rightarrow\ a_n = 0, \quad b_n = \begin{cases} \frac{8}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(f) $d_n = \frac{1}{2} C_n$

$$a_0 = 0, \quad a_n = 0, \quad b_n = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(g) $a_0 = C_0 = \frac{1}{4}$

$$a_n = 2\,\mathrm{Re}\{C_n\} = \begin{cases} 0 & n \text{ even} \\ -\frac{1}{\cancel{4}\pi} & n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} \dfrac{-1}{n\pi} & n \text{ even} \\ \dfrac{1}{n\pi} & n \text{ odd} \end{cases}$$

(h) $a_0 = \frac{1}{3}$, $a_n = \dfrac{-6\left[\cos \frac{2n\pi}{3} - 1\right]}{4\pi^2 n^2}$, $b_n = 0$

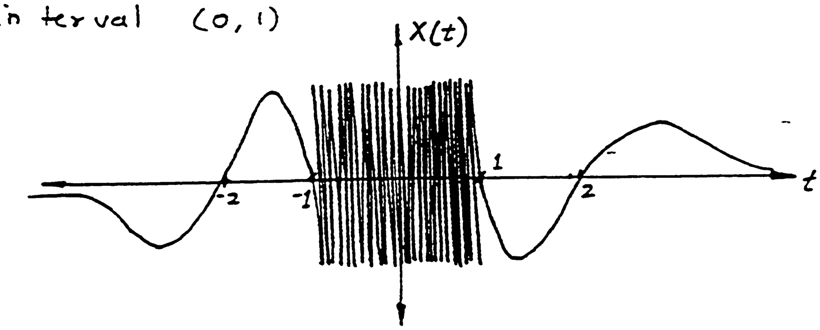
(i) $a_n = 0$

$$b_n = \frac{-2}{n\pi} + \frac{9}{2n^2\pi^2} \sin \frac{2n\pi}{3}$$

3.14(a) $C_n = \frac{1}{T} \int_0^T x(t) \exp\left[-j\frac{2n\pi t}{T}\right] dt$

$$|C_n| = \left| \frac{1}{T} \int_0^T x(t) \exp\left[-j\frac{2n\pi t}{T}\right] dt \right| \le \frac{1}{T} \int_0^T |x(t)|\, dt < \infty$$

(b) No it does not. The function $x(t) = \sin \frac{2\pi}{t}$ does not meet Dirichlet condition 2 As illustrated in the figure below the function has an infinite number of maxima and minima in the interval $(0, 1)$



(c) No it does not The function $x(t) = \tan 2\pi t$ does not meet Dirichlet 1, since $\int_0^{1/2} |x(t)|\, dt \to \infty$

3.15 $x(t) = t^2 \qquad -\pi < t \le \pi \qquad x(t+2\pi) = x(t)$

(a) $x(t) = x(-t) \Longrightarrow b_n = 0$

$$a_n = \frac{2}{T}\int_T x(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\, dt = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt\, dt$$

$$= \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2}\right] = 4\frac{\cos n\pi}{n^2}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{6\pi}t^3\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

(b) Set $t = 0 \qquad x(0) = 0$

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right)$$

or $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

3.16 $$c_n = \begin{cases} 0 & n = 0 \\ 1 + \exp[-jn\frac{\pi}{3}] - 2\exp[-j\frac{\pi}{n}] & n \ne 0 \end{cases}$$

Yes it does because

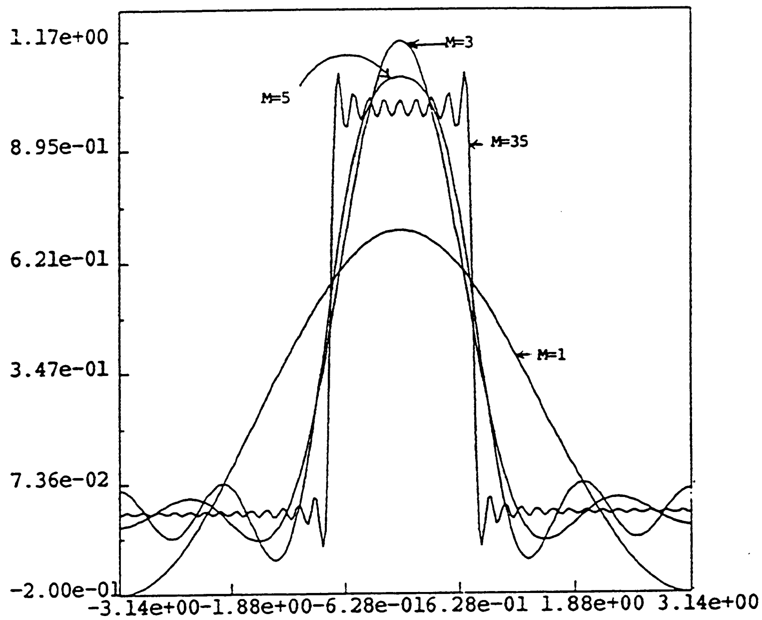
$$c_{-n} = 1 + \exp[j\frac{n\pi}{3}] - 2\exp[j\frac{\pi}{n}] = c_n^*$$

$$x(t) = \delta(t) + \delta(t - \frac{T}{6}) - 2\delta(t - \frac{T}{2})$$

3.17 $$x(t) = \frac{1}{4} + \sum_{n=1}^{M}\frac{2}{n\pi}\sin\frac{n\pi}{4}\cos 2n\pi t$$

$M = 1 \qquad x(t) = \dfrac{1}{4} + \dfrac{\sqrt{2}}{\pi}\cos 2\pi t$

(a)



(b) As $M \to \infty$, we predict that

$$x(t) = 1.00 \qquad |t| < \frac{\pi}{4}$$

$$= 0 \qquad \frac{\pi}{4} < |t| < \pi$$

3.18 (i) $C_n = \frac{1}{T} \int_T x(t) \exp[-j\frac{2\pi n t}{T}] dt$

$$= \int_{-1/2}^{1/2} \delta(t) \exp[-j 2\pi n t] dt = 1$$

$$\therefore \; x(t) = \delta(t) = \sum_{n=-\infty}^{\infty} \exp[j 2\pi n t]$$

(ii) $x(t) = -\delta(t) + \delta(t-1) \qquad x(t+2) = x(t)$

$$C_n = -\frac{1}{T} \int_{\langle T \rangle} x(t) \exp[-j\frac{2\pi n t}{T}] dt$$

$$= \frac{1}{2} \int_{-1/2}^{3/2} [-\delta(t) + \delta(t-1)] \exp[-j n\pi t] dt$$

$$= \frac{1}{2} [-1 + \exp[-jn\pi]]$$

$$= \begin{cases} -1 & n \text{ odd} \end{cases}$$

Thus,

$$x(t) = \sum_{n \text{ odd}} \exp[jn\pi] \exp[jn\pi t]$$

3.19 Suppose we consider a periodic waveform to have period NT. Then $\omega_0 = 2\pi/NT$ with Fourier coefficients

$$\hat{c}_n = \frac{1}{NT}\int_0^{NT} x(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{NT}\left[\int_0^{T} x(t)e^{-jn\omega_0 t}dt + \int_T^{2T} x(t)e^{-jn\omega_0 t}dt + \ldots + \int_{(N-1)T}^{NT} x(t)e^{-jn\omega_0 t}dt\right]$$

$$= \frac{1}{NT}\sum_{k=0}^{N-1}\int_{kT}^{(k+1)T} x(t)e^{-jn\omega_0 t}dt$$

Let $\tau = t - kT$. Then

$$\hat{c}_n = \frac{1}{NT}\sum_{k=0}^{N-1}\int_0^{T} x(\tau + kT)e^{-jn\omega_0(\tau+kT)}d\tau$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} e^{-jn\frac{2\pi}{NT}kT}\left[\frac{1}{T}\int_0^{T} x(\tau)e^{-jn\omega_0\tau}d\tau\right]$$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\right]c_n$$

where c_n are the Fourier coefficients of $x(t)$ considered as periodic with period T. Now

$$\frac{1}{N} = \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn} = \begin{cases} 1 & \text{if } n = pN,\ p \text{ an integer} \\ 0 & \text{otherwise} \end{cases}$$

so that

$\hat{c}_{pN} = c_n$ for $p = 0, \pm 1, \pm 2$, etc, all other $\hat{c}_n = 0$.

For signal (i), $\hat{c}_{3p} = \begin{cases} 1 & p = 0, \pm 1, \pm 2, \text{ etc} \\ 0 & \text{otherwise} \end{cases}$

For signal (ii), $\hat{c}_{3p} = \begin{cases} -1 & p = 1, \pm 3, \pm 5, \text{ etc} \\ 0 & \text{otherwise} \end{cases}$

3.20 (a) $c_5 = \left[\dfrac{\sin\dfrac{n\pi}{T}}{n\pi}\right]^2 = \dfrac{1}{T^2} = 150$, so that $T = \sqrt{150}$

(b) To find the average power in $x(t)$, we note from Equation (3.5.15) that we can condider $x(t)$ as the periodic convolution of the periodic square wave of Example 3.3.3 with itself, so that, with $K = 1$, $\tau = 1$, over the interval $[-T/2, T/2]$, we have

$$x(t) = \begin{cases} 1+t & -\dfrac{1}{2} \le t \le 0 \\ 1-t & 0 \le t \le \dfrac{1}{2} \end{cases}$$

Average power in $x(t)$ is $P = \dfrac{7}{12T} = 0.0476$

Average power in $\hat{x}(t)$ is $P = \sum_{n=-2}^{2} |c_n|^2 = 0.1592/\mathrm{T} = 0.0130$

3-21 (a) even symmetry

(b) odd symmetry

(c) half-wave odd symmetry

(d) odd symmetry

(e) half-wave even symmetry

$x_2(t) = y(t) + z(t)$

where $y(t) = x_1(t)$ with $\tau = 2$, $T = 4$

$z(t) = x_1(t)$ with $\tau = 3$, $T = 4$

3.22 $z(t) = \dfrac{1}{T}\displaystyle\int_{\langle T\rangle} x(\tau)\, y(t-\tau)\, d\tau$

(a) $z(t+T) = \dfrac{1}{T}\displaystyle\int_{\langle T\rangle} x(\tau)\, y(t+T-\tau)\, d\tau$

$= \dfrac{1}{T}\displaystyle\int_{\langle T\rangle} x(\tau)\, y(t-\tau)\, d\tau$

$\therefore$ $z(t)$ is periodic with period T

(b) $\frac{1}{T}\int_{T} x(\tau)\, y(t-\tau)\, d\tau = \frac{1}{T}\int_{T} x(t-r)\, y(r)\, dr$ (commutativity)

$$x(t) \circledast [y(t) \circledast v(t)] = \frac{1}{T}\int_{\langle T\rangle} x(\tau)\left[\frac{1}{T}\int_{\langle T\rangle} y(\sigma)\, v(t-\tau-\sigma)\, d\sigma\right] d\tau$$

$$= \frac{1}{T^2}\int_{\langle T\rangle}\int_{\langle T\rangle} x(\tau)\, y(\sigma)\, v(t-\tau-\sigma)\, d\sigma\, d\tau \quad (1)$$

$$[x(t) \circledast y(t)] \circledast v(t) = \frac{1}{T}\int_{\langle T\rangle}\left[\frac{1}{T}\int_{\langle T\rangle} x(\tau)\, y(r-\tau)\, d\tau\right] v(t-r)\, dr$$

$$= \frac{1}{T^2}\int_{\langle T\rangle}\int_{\langle T\rangle} x(\tau)\, y(r-\tau)\, v(t-r)\, dr \quad (2)$$

let $r-\tau = \sigma$ in (2), yields

$$[x(t) \circledast y(t)] \circledast v(t) = \frac{1}{T^2}\int_{\langle T\rangle}\int_{\langle T\rangle} x(\tau)\, y(\sigma)\, v(t-\tau-\sigma)\, d\sigma\, d\tau \quad (3)$$

$$= x(t) \circledast [y(t) \circledast v(t)] \quad \text{(associativity)}$$

3.23 The periodic convolution of $x(t)$ and $y(t)$ gives

$$z(t) = \begin{cases} \displaystyle\int_0^{t+\frac{1}{2}} (1-\tau)\, d\tau = -\frac{t^2}{2} + \frac{t}{2} + \frac{3}{8} & -\frac{1}{2} \le t \le \frac{1}{2} \\ \displaystyle\int_{t-\frac{1}{2}}^{1} (1-\tau)\, d\tau = \frac{t^2}{2} - \frac{3t}{2} + \frac{9}{8} & \frac{1}{2} \le t \le \frac{3}{2} \end{cases}$$

With $\omega_0 = \pi$, the Fourier coefficients of $z(t)$ are

$$\alpha_n = \frac{1}{2}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} [-\frac{t^2}{2} + \frac{t}{2} + \frac{3}{8}] e^{-j\pi n t}\, dt + \int_{\frac{1}{2}}^{\frac{3}{2}} [\frac{t^2}{2} - \frac{3t}{2} + \frac{9}{8}] e^{-j\pi n t}\, dt\right]$$

$$= \frac{1}{2}\left[\int_{\frac{-1}{2}}^{\frac{1}{2}} [(1-e^{-jn\pi})(\frac{t}{2} - \frac{t^2}{2}) + (\frac{3}{8} + \frac{1}{8}e^{-jn\pi}) dt] e^{-jn\pi t}\right]$$

For $n = 0$, we get $\alpha_0 = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}(\frac{3}{8}+\frac{1}{8})dt = \frac{1}{4}$

For $n \neq 0$, n even, $\alpha_n = \frac{1}{8}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-jn\pi t}dt = 0$

For n odd, $\alpha_n = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}[t^2+t+\frac{1}{4}]e^{-jn\pi t}dt$

which evaluates to

$$\alpha_n = \frac{2-jn\pi}{2n^3\pi^3}\sin\frac{n\pi}{2}$$

From Example 3.3.3, the coefficients of $x(t)$ are

$$\beta_n = \frac{\sin\frac{n\pi}{2}}{n\pi}$$

We find the coefficients for $y(t)$ as

$$\gamma_n = \frac{1}{2}\int_0^1 (1-t)e^{-jn\pi t}dt$$

$$= \begin{cases} \frac{1}{4} & n=0 \\ \frac{-j}{2n\pi} & n \text{ even, } n \neq 0 \\ \frac{1}{n^2\pi^2} - \frac{j}{2n\pi} & n \text{ odd} \end{cases}$$

It easily follows from the above set of equations that $\alpha_n = \beta_n\gamma_n$

3.24

(a) $x(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$, $c_0 = 0$

$$y(t) = \int x(t)\,dt = \sum_{n=-\infty}^{\infty} c_n \int \exp[jn\omega_0 t]\,dt$$

$$= \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0} \exp[jn\omega_0 t]$$

$$y(t+T) = \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0} \exp[jn\omega_0 (t+T)]$$

$$= \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0} \exp[jn\omega_0 t] = y(t)$$

(b) $d_n = \dfrac{c_n}{jn\omega_0}$ are smaller than the amplitudes of $x(t)$

(c) Integration deemphasize the high frequency components.

(d) The integrated waveform is smoother than the original waveform.

3.25 The signal in Figure P3.19 (b) can be written in terms of the signal in part (a) as

$$x_b(t) = 1 + x_a(2t - \pi/2)$$

$$= 1 + x_a(2[t - \pi/4])$$

therefore

$$d_n = 1 + c_n \exp[-j\tfrac{n\pi}{2}]$$

hence

$$x_a(t) = 1 - \frac{8}{\pi^2}\left(\cos 2t + \frac{1}{9}\cos 6t + \frac{1}{25}\cos 10t + \dots\right)$$

3.26 $C_n = \frac{1}{n^2+1} \exp[jn\frac{\pi}{3}]$

(a) $C_n^* = \frac{1}{n^2+1} \exp[-jn\frac{\pi}{3}] = C_{-n} \Rightarrow x(t)$ is real signal.

(b) Average value $= C_0 = 1$

(c) $RC \frac{dy(t)}{dt} + y(t) = x(t)$

or $\frac{dy}{dt} + y(t) = x(t)$

$H(\omega) = \frac{1}{j\omega+1}$ or $H(n\omega_0) = \frac{1}{1+jn\omega_0}$

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{1}{(1+jn\omega_0)} \frac{\exp[jn\pi/3]}{n^2+1} \exp[jn\omega_0 t]$$

$n = 0, \quad d_0 = C_0 = 1$

$n = 1 \quad d_1 = \frac{1}{2(1+j\omega_0)} \exp[j\frac{\pi}{3}]$

$n = 2 \quad d_2 = \frac{1}{5(1+j2\omega_0)} \exp[j2\frac{\pi}{3}]$

$d_3 = \frac{1}{10(1+j3\omega_0)} \exp[jn\pi]$

(d) for $\omega_0 \gg 1$,

the circuit deemphasize the high-frequency components

If $y(t)$ is the voltage across the resistor instead,

$H(\omega) = \frac{1}{1+\frac{1}{j\omega}} = \frac{j\omega}{1+j\omega}$; $\quad H(n\omega_0) = \frac{jn\omega_0}{1+jn\omega_0}$

$$y(n) = \sum_{n=-\infty}^{\infty} \frac{jn\omega_0}{1+jn\omega_0} \frac{\exp[jn\frac{\pi}{3}]}{1+n^2} \exp[jn\omega_0 t]$$

$$d_1 = \frac{j\omega_0}{2(1+j\omega_0)} \exp[j\tfrac{\pi}{3}]$$

$$d_2 = \frac{j2\omega_0}{5(1+j2\omega_0)} \exp[j\tfrac{2\pi}{3}]$$

$$d_3 = \frac{j3\omega_0}{10(1+j3\omega_0)} \exp[j\pi]$$

for $n\omega_0 >> 1$, there n^{th} harmonic can be written as

$$d_n \simeq \frac{1}{n^2+1} \exp[j\tfrac{n\pi}{3}] = c_n$$

i.e the circuit passes the high frequency components without alternations.

3.27

$$H(\omega) = \frac{\frac{1}{j\omega}}{1+\frac{1}{j\omega}} = \frac{1}{1+j\omega}$$

With $\omega_0 = 1$,

$x(t) = 1 + \frac{3}{2}e^{j30''}e^{j\omega_0 t} + \frac{3}{2}e^{-j30''}e^{-j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t}$, where

Therefore $c_0 = 1,\ c_1 = c_{-1}* = \frac{3}{2}e^{j30''},\ c_2 = c_{-2} = \frac{1}{2}$

$$H(0) = 1,\ H(\omega_0) = \frac{1}{1+j1} = 0.7071e^{-j45''},\ H(2\omega_0) = 0.4472e^{-j63.43''}$$

and

$$d_0 = 1,\ d_1 = d_{-1}* = 1.0607e^{-j15''},\ d_2 = d_{-2}* = 0.2236e^{-j63.43''}$$

which gives

$$y(t) = 1 + 2.1214\cos(t - 15'') + 0.4472\cos(2t - 63.43'')$$

3.28 System 1 has magnitude and phase distortion.

System 2 is distortionless.

System 3 has magnitude distortion.

System 4 has phase distortion.

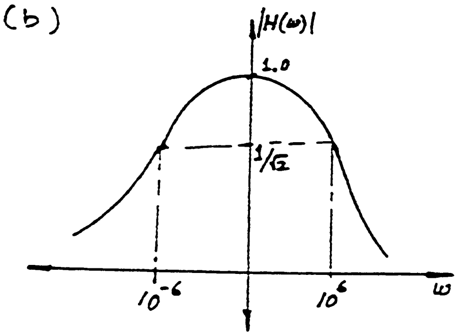
3.29 (a) For an input $x(t) = \exp[j\omega t]$, the output is

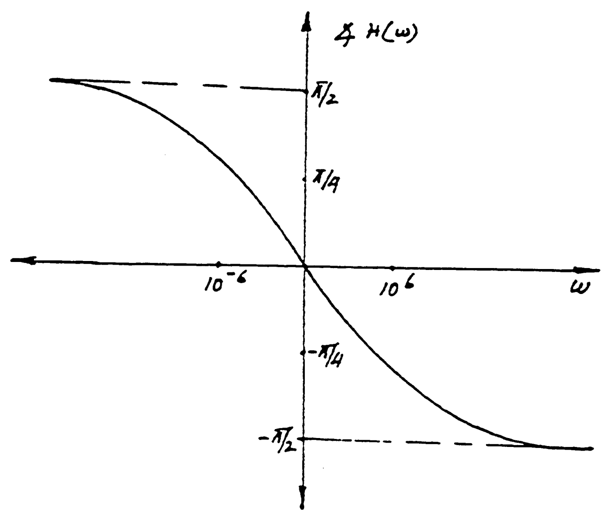
$$y(t) = \frac{\exp[j\omega t]}{R + \frac{1}{j\omega c}} \cdot \frac{1}{j\omega c}$$

$$= \frac{1}{1 + j\omega Rc} \exp[j\omega t] = H(\omega)\exp[j\omega t]$$

where $H(\omega) = \dfrac{1}{1 + j\omega Rc} = \dfrac{1}{1 + j10^{-6}\omega}$

(b)





(c) for $x(t) = 10\exp[j\omega t]$

$$y(t) = 10\,H(\omega)\exp[j\omega t]$$

$$\frac{|y(t) - x(t)|}{|x(t)|} = \frac{|10H(\omega) - 10|}{10} < 0.01$$

or $\quad |H(\omega) - 1.0| < 0.01$

$$\frac{|10^{-6}\omega|}{\sqrt{1 + 10^{-12}\omega^2}} < 0.01$$

$$\omega^2 < \frac{10^{-4}}{10^{-12} - 10^{-16}}$$

$$|\omega| < 10^4$$

(d) $\angle H(\omega) = -\tan^{-1} 10^{-6}\omega$

Need $\dfrac{|\tan^{-1} 10^{-6}\omega - 10^{-6}\omega|}{|10^{-6}\omega|} < 0.02$

which gives $\omega < 249450$

3.30 $y(t) = A\,x(t) + B\,x^2(t)$

$x(t) = a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t$

$y(t) = A[a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t] + B[a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t]^2$

$= Aa_1 \cos\omega_0 t + Aa_2 \cos 2\omega_0 t + B[a_1^2 \cos^2\omega_0 t$

$+ 2a_1 a_2 \cos\omega_0 t \cos 2\omega_0 t + a_2^2 \cos^2 2\omega_0 t]$

$= Aa_1 \cos\omega_0 t + Aa_2 \cos 2\omega_0 t + B\Big[a_1^2 \dfrac{1 + \cos 2\omega_0 t}{2}$

$+ 2a_1 a_2 \left(\dfrac{\cos\omega_0 t + \cos 3\omega_0 t}{2}\right) + \dfrac{a_2^2}{2}(1 + \cos 4\omega_0 t)\Big]$

$= Aa_1 \cos\omega_0 t + Aa_2 \cos 2\omega_0 t + \dfrac{Ba_1^2}{2} + \dfrac{B\cos 2\omega_0 t}{2}$

$+ Ba_1 a_2 \cos\omega_0 t + Ba_1 a_2 \cos 3\omega_0 t + \dfrac{Ba_2^2}{2} + \dfrac{Ba_2^2}{2}\cos 4\omega_0 t$

The new harmonics are $Ba_1 a_2 \cos 3\omega_0 t$,

$\dfrac{Ba_2^2}{2}\cos 4\omega_0 t$

Also a DC term with magnitude $\dfrac{Ba_1^2 + Ba_2^2}{2}$ was generated.

3.31 Now $f_0 = 2$ kHz and $c_n = (K\tau/T)\text{sinc}(n\tau/T) = (1/5)\text{sinc}(n/5)$

The filter passes only the dc, 1st and 2nd harmonics.

For $0 \le |f| \le 5$ kHz, $|H(f)| = 10$, Arg $H(f) = -(\pi f)/10000$ rads.

Now $c_0 = \frac{1}{5}$, $c_1 = c_{-1}* = \frac{1}{5}\text{sinc}(\frac{1}{5})$, $c_2 = c_{-2}* = \frac{1}{5}\text{sinc}(\frac{2}{5})$

so that

$$d_0 = 2,\ d_1 = d_{-1}* = 2\text{sinc}(\frac{1}{5})e^{-j\frac{\pi}{5}},\ d_2 = d_{-2}* = 2\text{sinc}(\frac{2}{5})e^{-j\frac{2\pi}{5}}$$

and

$$y(t) = 2 + \text{sinc}(\frac{1}{5})\cos(4000\pi t - 36^o) + \text{sinc}(\frac{2}{5})\cos(8000\pi t - 72^o)$$

3.32 From Example 3.4.2, $a_n = \begin{cases} \dfrac{8A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

With $b_n = 0$, we get $c_n = \begin{cases} \dfrac{4A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

For the given circuit, $H(\omega) = 1/(R + j\omega L) = 1/(100 + j0.5\pi)$

so that with $\omega_0 = \pi/2$, we have

$$d_n = \frac{1}{100 + j0.5\pi n} \text{ x } \frac{40}{\text{n}\pi} \quad \text{n odd}$$

and

$$|d_n| = \frac{1}{(100)^2 + (0.5\pi n)^2} \text{ x } \frac{1600}{(\text{n}\pi)^2} \quad \text{n odd}$$

The average power dissipated in the resistor is

$$P = \sum_{n \text{ odd}} |d_n|^2 \simeq 0.018$$

3.33 From Example 3.3.2

$$c_n = \begin{cases} \dfrac{E}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd}, n \neq \pm 1 \end{cases}$$

$$c_1 = c_{-1}{}^* = \frac{E}{4j}$$

Since $H(\omega) = 2/(j\omega + 3)$,

$$d_0 = \frac{2E}{3\pi}$$

$$d_1 = d_{-1}{}^* = -j\frac{E}{4} \times \frac{2}{j\omega_0 + 3} = \frac{E}{2\sqrt{9+\omega_0^2}} \quad -\frac{\pi}{2} - \tan^{-1}\frac{\omega_0}{3}$$

$$d_2 = d_{-2}{}^* = -\frac{E}{3\pi} \times \frac{2}{j2\omega_0 + 3} = \frac{2E}{3\sqrt{9+4\omega_0^2}} \quad -\pi - \tan^{-1}\frac{2\omega_0}{3}$$

3.34 With $H(\omega) = \dfrac{j\omega + 1}{(j\omega)^2 + 3j\omega + 2} = \dfrac{1}{j\omega + 2}$, we have

$$d_0 = \frac{2E}{3\pi}$$

$$d_1 = d_{-1}{}^* = \frac{E}{2\sqrt{4+\omega_0^2}} \angle\left(-\frac{\pi}{2} - \tan^{-1}\frac{\omega_0}{2}\right)$$

$$d_2 = d_{-2}{}^* = \frac{2E}{6\sqrt{1+\omega_0^2}} \angle\left(-\pi - \tan^{-1}\omega_0\right)$$

3.35 $y_c(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)\cos n\omega_0\tau \, h(t-\tau)\, d\tau = \frac{1}{T_1}\int_{t-T_1}^{t} x(\tau)\cos n\omega_0\tau \, d\tau$

$y_s(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)\sin n\omega_0\tau \, h(t-\tau)\, d\tau = \frac{1}{T_1}\int_{t-T_1}^{t} x(\tau)\sin n\omega_0\tau \, d\tau$

Also $c_n = \displaystyle\frac{1}{T}\int_0^T x(\tau)\exp[-jn\omega_0\tau]\, d\tau$

$= \displaystyle\frac{1}{T}\int_0^T x(\tau)\cos n\omega_0\tau\, d\tau - j\frac{1}{T}\int_0^T x(\tau)\sin n\omega_0\tau\, d\tau$

$\Rightarrow y_c(t) - j\, y_s(t)$

(a) $T_1 = T$ $\quad y_c(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\cos n\omega_0\tau\, d\tau$

$$= \frac{1}{T}\int_0^T x(\tau)\cos n\omega_0\tau\, d\tau = Re\{C_n\}$$

$$y_s(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\sin n\omega_0\tau\, d\tau = \frac{1}{T}\int_0^T x(\tau)\sin n\omega_0\tau\, d\tau$$

$$= -Im\{C_n\}$$

(b) $T_1 = lT$,

$$y_c(t) = \frac{1}{lT}\int_{t-lT}^{t} x(\tau)\cos n\omega_0\tau\, d\tau = \frac{l}{lT}\int_0^T x(\tau)\cos n\omega_0\tau\, d\tau$$

$$= \frac{1}{T}\int_0^T x(\tau)\cos n\omega_0\tau\, d\tau$$

$$= Re\{C_n\}$$

$$y_s(t) = \frac{1}{lT}\int_{t-lT}^{t} x(\tau)\sin n\omega_0\tau\, d\tau$$

$$= \frac{1}{lT}\int_0^T x(\tau)\sin n\omega_0\tau\, d\tau = \frac{1}{T}\int_0^T x(\tau)\sin n\omega_0\tau\, d\tau$$

$$= -Im\{C_n\}$$

(c) $T_1 \gg T$ but $T_1 \neq lT$

assume $T_1 = lT + fT$ $\quad l =$ integer, $0 < f < 1$ and $l \gg f$

$$y_c(t) = \frac{1}{T_1}\int_{t-T_1}^{t} x(\tau)\cos n\omega_0\tau\, d\tau = \frac{1}{lT+fT}\int_{t-T_1}^{(t-T_1)+lT} x(\tau)\cos n\omega_0\tau\, d\tau$$

$$+ \frac{1}{lT+fT}\int_{(t-T_1)+lT}^{(t-T_1)+lT+fT} x(\tau)\cos n\omega_0\tau\, d\tau$$

Since $lT \gg fT$ $\quad y_c(t) \simeq \frac{1}{lT}\int_0^{lT} x(\tau)\cos n\omega_0\tau\, d\tau$

$$+ \frac{1}{lT}\int_0^{fT} x(\tau)\cos n\omega_0\tau\, d\tau$$

$$\simeq \frac{1}{T}\int_0^T x(\tau)\cos n\omega_0\tau\, d\tau = Re\{C_n\}$$

Similarly $\quad y_s(t) \simeq -Im\{C_n\}$

3.36 $H(\omega) = \dfrac{R_2}{(R_1 + R_2) + j\omega R_1 R_2 C}$

$= \dfrac{1}{2 + j5\omega \times 10^{-2}}$

$|d_2| = |C_2 H(2\omega_o)|$, with $\omega_o = 1$

$= \left| \dfrac{1}{\pi(1-4)} \cos\pi \dfrac{1}{2 + j\,10 \times 10^{-2}} \right| = 0.05298$

$|d_4| = |C_4 H(4\omega_o)|$

$= \left| \dfrac{1}{\pi(1-16)} \cos 2\pi \dfrac{1}{2 + j\,20 \times 10^{-2}} \right| = 0.01056$

3.37 $H(\omega) = \dfrac{R}{(R - \omega^2 LRC) + j\omega L}$

$= \dfrac{1}{(1 - 10^{-5}\omega^2) + j\,10^{-4}\omega}$

$|d_2| = |C_2 H(2\omega_o)|$, with $\omega_o = 1$

$= \left| \dfrac{1}{\pi(1-4)} \cos\pi \dfrac{1}{(1 - 10^{-5}) + j\,10^{-4}} \right| = 0.1061$

$|d_4| = |C_4 H(4\omega_o)|$

$= \left| \dfrac{1}{\pi(1-16)} \cos 2\pi \dfrac{1}{(1 - 4 \times 10^{-5}) + j\,2 \times 10^{-4}} \right| = 0.02122$

3.38 (a) $c_n = \begin{cases} \dfrac{2}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

For the circuit of Figure 3.36, since $H(\omega) = 20/(40 + j\omega)$

$$d_0 = \frac{1}{\pi}, \quad |d_2| = \frac{40}{3\pi\sqrt{600 + \omega_0^2}}$$

For the circuit of Figure 3.37, since

$$H(\omega) = \frac{10^4}{10^4 - 0.1\omega^2 + j\omega}$$

$$d_0 = \frac{2}{\pi}, \quad |d_2| = \frac{2\text{x}10^4}{3\pi\sqrt{(10^4 - 0.1\omega^{2)})^2 + \omega^2}}$$

(b) We have

$$c_n = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

For the system of Figure 3.36, we get

$$d_1 = d_{-1}{}^* = \frac{800}{\pi(40 + j\omega_0)}$$

For the system of Figure 3.36, we get

$$d_1 = d_{-1}{}^* = \frac{40\text{x}10^4}{\pi(10^4 - 0.1\omega^2 + j\omega)}$$

3.39 (a) $$\sum_{n=-N}^{N} \exp[jn\omega_0 t] = \frac{\exp[-jN\omega_0 t] - a^{j(N+1)\omega_0 t}}{1-\exp[j\omega_0 t]}$$

$$= \frac{\exp\left[j\frac{\omega_0 t}{2}\right]\left[\exp[-j(N+\tfrac{1}{2})\omega_0 t - a^{j(N+\frac{1}{2})\omega_0 t}\right]/2j}{\exp\left[j\frac{\omega_0 t}{2}\right]\left[\exp[-j\frac{\omega_0 t}{2}] - \exp[j\frac{\omega_0 t}{2}]\right]/2j}$$

$$= \frac{\sin\left[(N+\frac{1}{2})\omega_0 t\right]}{\sin\frac{\omega_0 t}{2}}$$

(b) $$\frac{1}{T}\int_{-T/2}^{T/2} \frac{\sin(N+\frac{1}{2})\omega_0 t}{\sin(\omega_0 t/2)}\,dt \quad , \quad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-N}^{N} \exp[jn\omega_0 t]\,dt = \frac{1}{T}\sum_{n=-N}^{N}\int_{-T/2}^{T/2} \exp[jn\omega_0 t]\,dt$$

$$= \frac{1}{T}\sum_{n=-N}^{N} \left.\frac{\exp[jn\omega_0 t]}{jn\omega_0}\right|_{-\frac{T}{2}}^{T/2} ,$$

$$= \sum_{n=-N}^{N}\left(\frac{\exp[jn\pi]-\exp[-jn\pi]}{j2\pi n}\right)$$

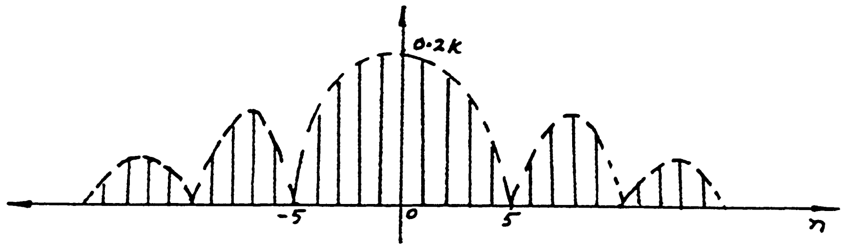
note for $n \neq 0$, $$\frac{\exp[jn\pi]-\exp[-jn\pi]}{j2\pi n} = 0$$

$$\therefore \frac{1}{T}\int_{-T/2}^{T/2} \frac{\sin(N+\frac{1}{2})\omega_0 t}{\sin(\omega_0 t/2)}\,dt = \sum_{n=-N}^{N}\left(\frac{\exp[jn\pi]-\exp[-jn\pi]}{j2\pi n}\right)$$

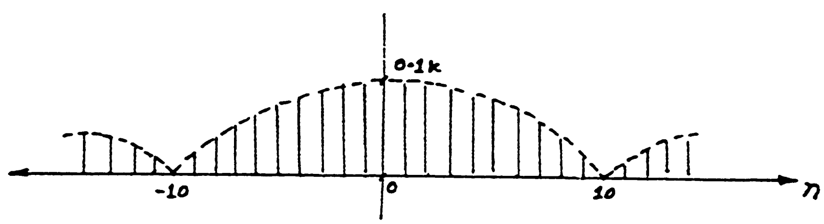
$$= 1$$

3.40 $C_n = \frac{K\tau}{T} \operatorname{sinc}(n\pi\, \tau/T)$ (from 3.2.15)

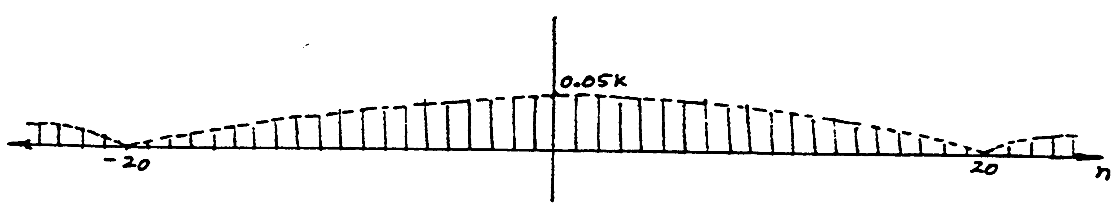
(i) $\frac{\tau}{T} = 0.2$ $|C_n| = 0.2\, K \operatorname{sinc}(0.2 n\pi)$



(ii) $\frac{\tau}{T} = 0.1$ $|C_n| = 0.1\, K \operatorname{sinc}(0.1 n\pi)$



(iii) $\frac{\tau}{T} = 0.05$ $|C_n| = 0.05\, K \operatorname{sinc}(0.05 n\pi)$



Conclusions :

(1) the magnitude of the spectrum increases proportional to τ,

(2) the frequency content of the signal is compressed when τ increases.

3.41 $C_n = \frac{k}{n\pi} \sin\left(\frac{n\pi\tau_o}{T}\right)$

$$\tilde{C}_2 = \frac{k}{2\pi} \sin\left(\frac{2\pi\tau}{T}\right), \qquad \tau = \tau_o - \epsilon$$

$$= \frac{k}{2\pi} \sin\left(\frac{2\pi}{T}(\tau_o - \epsilon)\right)$$

$$= \frac{k}{2\pi} \sin\left(\frac{2\pi}{T}\tau_o - \frac{2\pi\epsilon}{T}\right) = \frac{k}{2\pi} \sin\frac{2\pi}{T}\tau_o \cos\frac{2\pi\epsilon}{T}$$

$$- \frac{k}{2\pi} \cos\frac{2\pi}{T}\tau_o \sin\frac{2\pi\epsilon}{T}$$

when $\epsilon \ll \tau_o$ i.e. $\epsilon \to 0$

$$\sin\frac{2\pi\epsilon}{T} \simeq \frac{2\pi\epsilon}{T}, \qquad \cos\frac{2\pi\epsilon}{T} \simeq 1$$

$$\tilde{C}_2 \simeq \frac{k}{2\pi} \sin\frac{2\pi}{T}\tau_o - \frac{k}{2\pi}\,\frac{2\pi\epsilon}{T} \cos\frac{2\pi}{T}\tau_o$$

$$= \frac{k}{2\pi} \sin\frac{2\pi}{T}\tau_o - \frac{k\epsilon}{T} \cos\frac{2\pi}{T}\tau_o$$

At $T = 10$, $\tau = 1$ $\epsilon = 0.1$

C_2 exact $= \frac{k}{2\pi} \sin\left(\frac{\pi}{5}\right) = 0.0935\,k$

$$\% \text{ change} = \left| \frac{\tilde{C}_2 - C_2}{C_2} \right| \times 100$$

$$= \left| \frac{\frac{k\epsilon}{T} \cos\frac{2\pi}{T}\tau_o}{\frac{k}{2\pi} \sin\frac{2\pi}{T}\tau_o} \right| \times 100$$

$$= 100 \left| \frac{\epsilon}{2\pi} \cot\frac{2\pi}{T}\tau_o \right|$$

$$= 100 \times \frac{0.1}{2\pi} \times 1.37638$$

$$= 2.19\,\%$$

3.42 (a)

$$C_n = \begin{cases} \frac{1}{\pi}\left[A\left(\frac{\sin(n-1)t_0}{n-1} - \frac{\sin(n+1)t_0}{n+1}\right) + \frac{2B\cos nt_0}{n}\right], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(b) $$|C_3| = \frac{A}{12\pi}\left|2\sin 2t_0 + \sin 4t_0\right|$$

when $B = A/2$, $t_0 = 0.5236$ rad.

in this case

$$|C_3| = 0.068916\ A$$

(c) $|C_3|$ is maximum at $t_0 = 30^\circ$ and 150°

3.43 (a)

$$C_n = \begin{cases} \frac{-A}{2\pi}\left[\frac{2}{n^2-1} + \frac{\exp[-j(n+1)t_0]}{n+1} - \frac{\exp[-j(n-1)t_0]}{n-1}\right], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(b) $|C_3|$ is maximum at $t_0 = 2.234$ rad (128°)

$$|C_3|_{max} = 0.99691\ A$$

(c) The method of generating the 3rd harmonic in problem 3.37 is more efficient.

3.44 (a) $\omega t_1 = \pi - \tan^{-1}(2\pi \times 6.0)$

$= 1.5973$

$$A = \sin(1.5973)\exp\left[\frac{1.5973}{37.7}\right]$$

$= 1.0429$

$$A\exp\left[\frac{-t_2}{RC}\right] = 0.85895 = \sin\omega t_2$$

(b) $$c_n = \int_{\pi/2\omega}^{t_2} A\exp[-10t]\exp[-jnt]\,dt + \int_{t_2}^{5\pi/2\omega} \sin t\,\exp[-jnt]\,dt$$

$$= \frac{A}{10+jn}\left[\exp\left[-(10+jn)\frac{\pi}{2\omega}\right] - \exp\left[-(10+jn)t_2\right]\right]$$

$$+ \frac{1}{2(n-1)}\left[\exp\left[-j(n-1)\frac{5\pi}{2\omega}\right] - \exp\left[-j(n-1)t_2\right]\right]$$

$$- \frac{1}{2(n+1)}\left[\exp\left[-j(n+1)\frac{5\pi}{2\omega}\right] - \exp\left[-j(n+1)\frac{t}{2}\right]\right]$$

(c) $$c_0 = 1.472 \times 10^{-2}$$

$$|c_1| = 1.3561 \times 10^{-2}$$

$$\frac{|c_1|}{c_0} = 0.92126$$

CHAPTER 4

4.1 (a) $x(-t) \longleftrightarrow X(-\omega)$

(b) $x_e(t) = (x(t) + x(-t))/2 \longleftrightarrow (X(\omega) + X(-\omega))/2$

(c) $x_o(t) = (x(t) - x(-t))/2 \longleftrightarrow (X(\omega) - X(-\omega))/2$

(d) $x^*(t) \longleftrightarrow X^*(-\omega)$

(e) $\text{Re}\{x(t)\} = (x(t) + x^*(t))/2 \longleftrightarrow (X(\omega) + X^*(-\omega))/2$

(f) $\text{Im}\{x(t)\} = (x(t) - x^*(t))/2j \longleftrightarrow (X(\omega) - X^*(-\omega))/2j$

4.2 (a) $x(t) = e^{-2t}u(-t)$ is not absolutely integrable.
$X(\omega)$ does not exist

(b) $x(t) = |t|u(t) = t\,u(t)$ is not absolutely integrable.
No Fourier transform

(c) $x(t) = \cos\frac{\pi}{t}$ is not well behaved at $t = 0$. No $X(\omega)$.

(d) $x(t) = \frac{1}{t}$ is not absolutely integrable. No $X(\omega)$

(e) $x(t) = t^2 e^{-2t}u(t)$ is absolutely integrable.
$X(\omega)$ exists.

4.3
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp[-j\omega t]\,dt$$
$$= \int_{-\infty}^{\infty} x(t) \sum_{n=0}^{\infty} \frac{[-j\omega t]^n}{n!}\,dt$$
$$= \sum_{n=0}^{\infty} \frac{(-j\omega)^n}{n!} \int_{-\infty}^{\infty} t^n x(t)\,dt$$
$$= \sum_{n=1}^{\infty} (-j)^n m_n \frac{\omega^n}{n!}$$

4.4 $\int_{-\infty}^{\infty} \cos\omega t\, d\omega = \lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} \cos\omega t\, d\omega = \lim_{\Omega\to\infty} \frac{2\sin\Omega t}{t} = 2\pi\delta(t)$

4.5 (a) $\mathcal{F}\{x(-t)\} = X(-\omega) = rect\left[\frac{\omega+1}{2}\right]$

(b) $\mathcal{F}\{t\, x(t)\} = j\frac{dX(\omega)}{d\omega} = j\left[\delta(\omega) - \delta(\omega-2)\right]$

(c) $\mathcal{F}\{x(t+1)\} = X(\omega)e^{j\omega} = e^{j\omega}\, rect\left[\frac{\omega-1}{2}\right]$

(d) $\mathcal{F}\{x(-2t+4)\} = \mathcal{F}\{x[-2(t-2)]\} = \frac{1}{2}e^{-j2\omega}X(-\frac{\omega}{2})$

$= \frac{1}{2}e^{-j2\omega}\, rect\left[\frac{\omega+2}{4}\right]$

(e) $\mathcal{F}\{(t-1)x(t+1)\} = \mathcal{F}\{(t+1)x(t+1) - 2x(t-1)\}$

$= j\, e^{j\omega}\frac{dX(\omega)}{d\omega} - 2e^{j\omega}X(\omega)$

$= j\,\delta(\omega) - j\, e^{j\omega}\delta(\omega-2) - 2e^{j\omega}\, rect\left(\frac{\omega-1}{2}\right)$

(f) $\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega) = j\omega\, rect\left(\frac{\omega-1}{2}\right)$

(g) $\mathcal{F}\left\{t\frac{dx(t)}{dt}\right\} = j\frac{d[j\omega X(\omega)]}{d\omega}$

$= -\omega\frac{dX(\omega)}{d\omega} - X(\omega)$

$= 2\delta(\omega-2) - rect\left(\frac{\omega-1}{2}\right)$

(h) $\mathcal{F}\{x(2t-1)\} = \mathcal{F}\{x(2(t-\frac{1}{2}))\} = \mathcal{F}\{x(2t)\}e^{-j\omega/2}$

$= \frac{1}{2}e^{-j\omega/2}\, rect\left[\frac{\omega-2}{4}\right]$

$\therefore \mathcal{F}\{x(2t-1)\, e^{-j2t}\} = \frac{1}{2}e^{-j\frac{(\omega+2)}{2}}\, rect\,\frac{\omega}{4}$

(i) $\mathcal{F}\{x(t)e^{-j2t}\} = X(\omega+2) = rect\left(\frac{\omega+1}{2}\right)$

(j) $\mathcal{F}\{t\, x(t)\, e^{-j2t}\} = j\frac{d\,\mathcal{F}\{x(t)e^{-j2t}\}}{d\omega}$

$= j\frac{d}{d\omega}\, rect\left(\frac{\omega+1}{2}\right) = j\left[\delta(\omega+2) - \delta(\omega)\right]$

(k) $\mathcal{F}\{(t-1)x(t-1)\,e^{-j2t}\} = \mathcal{F}\{(t-1)x(t-1)e^{-j2(t-1)}e^{-j2}\}$

$$= e^{-j2}\,e^{-j\omega}\;j\,[\delta(\omega+2) - \delta(\omega)]$$

(l) $\mathcal{F}\{\int_{-\infty}^{t} x(\tau)\,d\tau\} = \dfrac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$

$$= \frac{1}{j\omega}\,\mathrm{rect}\left(\frac{\omega-1}{2}\right) + \frac{\pi}{2}\,\delta(\omega)$$

4.6 $\quad X(\omega) = \dfrac{1}{2+j\omega}$

$$\therefore Y(\omega) = \frac{e^{-j\omega} + e^{j\omega}}{2+j\omega} = \frac{2\cos\omega}{2+j\omega}$$

4.7 From (4.2.14), $\int_{-\infty}^{\infty} \exp[j\omega_0 t]\exp[-j\omega t]\,dt = 2\pi\delta(\omega-\omega_0)$

If we let $\omega_0 = 1$, $\omega = -\Omega$, we get $\int_{-\infty}^{\infty}\exp[j\Omega t]\,dt = 2\pi\delta(-\Omega)$ $= 2\pi\delta(\Omega)$

Replacing Ω by ω and using Euler's rule gives

$$\int_{-\infty}^{\infty}[\cos\omega t + j\sin\omega t]\,dt = 2\pi\,\delta(\omega)$$

You can think of the integral above as the sum of an infinite number of sines and cosines of various frequencies. At any t, except $t=0$, as many of these terms will be positive as will be negative and hence their contributions will add up to zero. At $t=0$ however, all the cosine terms are $+1$, and their contributions add up to produce a spike.

4.8 $\quad x(t) = \mathrm{rect}\left((t+\frac{a}{2})/a\right) - \mathrm{rect}\left((t-\frac{a}{2})/a\right)$

$$X(\omega) = a\,\mathrm{sinc}\frac{\omega a}{2\pi}\exp\left[j\frac{a\omega}{2}\right] - a\,\mathrm{sinc}\frac{\omega a}{2\pi}\exp\left[-j\frac{a\omega}{2}\right]$$

$$= a\,\mathrm{sinc}\frac{\omega a}{2\pi}\left[2j\sin\frac{\omega a}{2}\right]$$

Since $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$, and $x(0) = 0$,

$$Y(\omega) = \frac{X(\omega)}{j\omega} = \frac{2a}{\omega} \operatorname{sinc}\left(\frac{\omega a}{2\pi}\right) \sin\left(\frac{\omega a}{2}\right)$$

4.9 (a) $X(\omega) = \dfrac{1}{2 + j\omega}$; $|X(\omega)|^2 = \dfrac{1}{4 + \omega^2}$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4 + \omega^2}\, d\omega = \frac{1}{2\pi} \tan^{-1} \frac{\omega}{2} \Big|_0^{\infty} = \frac{1}{4}$$

(b) $X(\omega) = \displaystyle\int_0^5 e^{-j\omega t}\, dt = \frac{1 - e^{-j\omega 5}}{j\omega}$

$$= e^{-j\frac{5}{2}\omega} \cdot \frac{2 \sin\left(\frac{5\omega}{2}\right)}{j\omega}$$

$$\therefore |X(\omega)|^2 = \frac{4 \sin^2\left(\frac{5\omega}{2}\right)}{\omega^2} = \frac{2(1 - \cos 5\omega)}{\omega^2}$$

$$\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(1 - \cos 5\omega)}{\omega^2}\, d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos(5\omega)}{\omega^2}\, d\omega = \frac{2}{\pi} \cdot \frac{5\pi}{2} = 5$$

(c) $x(t) = \Delta\left(\dfrac{t}{4}\right) = \dfrac{1}{4} \operatorname{rect}\left(\dfrac{t}{4}\right) * \operatorname{rect}\left(\dfrac{t}{4}\right)$

$$\therefore X(\omega) = \frac{1}{4}\left(\frac{2 \sin \omega}{\omega}\right)^2 = \frac{\sin^2 \omega}{\omega^2} = \frac{1 - \cos 2\omega}{2\omega^2}$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1 - \cos 2\omega}{2\omega^2}\right]^2 d\omega = 4$$

(d) $X(\omega) = \operatorname{rect}\left(\dfrac{\omega}{2\pi}\right)$

$$\therefore E = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1$$

4.10 $x(t) = A\exp[-at^2] \qquad y(t) = x^2(t)$

(a) $X(\omega) = A\sqrt{\frac{\pi}{a}}\,\exp[-\omega^2/4a]$

$$y(t) = x^2(t) \longleftrightarrow Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(A\sqrt{\frac{\pi}{a}}\right)\exp\left[-\frac{\tau^2}{4a}\right]\left(A\sqrt{\frac{\pi}{a}}\right)\exp\left[-\frac{(\omega-\tau)^2}{4a}\right]d\tau$$

$$= \frac{A}{2a}\,\exp\left[-\frac{\omega^2}{8a}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{(\tau-\frac{\omega}{2})^2}{2a}\right]d\tau$$

using $\displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\,\exp[-x^2/2\sigma^2]\,dx = 1$

$$Y(\omega) = \frac{A}{2a}\cdot\exp[-\omega^2/8a].\sqrt{2\pi a} = A\sqrt{\frac{\pi}{2a}}\,\exp[-\omega^2/8a]$$

(b) $z(t) = y(t) * h(t) \implies Z(\omega) = Y(\omega)\,H(\omega)$

$$Z(\omega) = A\sqrt{\frac{\pi}{2a}}\,\exp[-\omega^2/8a]\cdot B\sqrt{\frac{\pi}{a}}\,\exp[-\omega^2/4b]$$

$$= \frac{AB\pi}{\sqrt{2ab}}\,\exp\left[-\omega^2\left(\frac{1}{8a}+\frac{1}{4b}\right)\right]$$

$$z(t) = \frac{AB\pi}{\sqrt{2ab}}\cdot\sqrt{\frac{ab}{\pi(a+b/2)}}\quad\exp\left[-\frac{ab}{a+b/2}\cdot t^2\right]$$

$$= AB\sqrt{\frac{\pi}{2a+b}}\;\exp\left[-\frac{ab}{a+b/2}\;t^2\right]$$

(c) $$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{(\;)^2} \longrightarrow z(t)$$

$$Y(\omega) = H(\omega)\,X(\omega) = AB\sqrt{\frac{\pi^2}{ab}}\left(\exp[-\omega^2/4a - \omega^2/4b]\right)$$

$$= AB\sqrt{\frac{\pi^2}{ab}}\;\exp\left[-\frac{(a+b)}{4ab}\,\omega^2\right]$$

$$y(t) = AB\sqrt{\frac{\pi}{a+b}}\;\exp\left[-\frac{ab}{a+b}\,t^2\right]$$

$$z(t) = (AB)^2\frac{\pi}{a+b}\;\exp\left[\frac{-2ab}{a+b}\,t^2\right]$$

4.11 (a) let $x(t) = \text{rect}\,(t/2) \longleftrightarrow X(\omega) = \dfrac{2\sin\omega}{\omega}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\omega}{\omega}\exp[j\omega t]\,d\omega$$

$$x(0) = 1 = \frac{2}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}\,d\omega \quad \text{or} \quad \int_{-\infty}^{\infty}\frac{\sin\omega}{\pi\omega} = 1$$

(b) $x(t) = \exp[-t^2/4\pi] \longleftrightarrow X(\omega) = 2\pi\exp[-\pi\omega^2]$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(2\pi\exp[-\pi\omega^2]\,\exp[j\omega t]\,d\omega$$

$$x(0) = 1 = \int_{-\infty}^{\infty}\exp[-\pi\omega^2]\,d\omega$$

(c) $x(t) = \exp[-a|t|] \longleftrightarrow X(\omega) = \dfrac{2a}{a^2+\omega^2}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2a}{a^2+\omega^2}\right)\exp[j\omega t]\,d\omega$$

$$x(0) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+\omega^2}\,d\omega$$

(d) $x(t) = \Delta(t/T) \longleftrightarrow X(\omega) = T\left(\dfrac{\sin\omega T/2}{\omega T/2}\right)^2$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}T\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2\exp[j\omega t]\,d\omega$$

$$x(0) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}T\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2 d\omega$$

4.12 (a) $x(t) = y(t) = \dfrac{1}{a^2+t^2}$, $X(\omega) = Y(\omega) = \dfrac{\pi}{a}\exp[-a|\omega|]$

$$\int_0^{\omega} \left(\frac{1}{a^2+t^2}\right)^2 dt = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1}{a^2+t^2}\right)^2 dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{\pi}{a}\right)^2 \exp[-2a|\omega|]\, d\omega$$

$$= \frac{\pi}{4a^2}\cdot\frac{1}{a} = \frac{\pi}{4a^3}$$

(b) $x(t) = \dfrac{1}{a^2+t^2}$, $X(\omega) = \dfrac{\pi}{a}\exp[-a|\omega|]$

$y(t) = \sin ct$, $Y(\omega) = \mathrm{rect}(\omega/2\pi)$

$$\int_0^{\infty}\frac{\sin ct}{a^2+t^2}\, dt = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin ct}{a^2+t^2}\, dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\pi}{a}\exp[-a|\omega|]\,\mathrm{rect}(\omega/2\pi)\, d\omega$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\pi}{a}\exp[-a|\omega|]\, d\omega$$

$$= \frac{1-\exp[-\pi a]}{2a^2}$$

(c) $x(t) = \dfrac{\sin^2 t}{t^2}$, $X(\omega) = \pi\,\Delta(\omega/2)$

$Y(\omega) = \dfrac{\sin t}{t}$, $Y(\omega) = \pi\,\mathrm{rect}(\omega/2)$

$$\int_{-\infty}^{\infty}\frac{\sin^3 t}{t^3}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi^2\,\Delta(\omega/2)\,\mathrm{rect}(\omega/2)\, d\omega$$

$$= \frac{\pi}{2}\int_{-1}^{1}\Delta(\omega/2)\, d\omega$$

$$= \frac{3\pi}{4}$$

(d) $x(t) = \dfrac{\sin^2 t}{t^2}$, $X(\omega) = \pi\, \Delta(\omega/2)$

$y(t) = \dfrac{\sin^2 t}{t^2}$, $Y(\omega) = \pi\, \Delta(\omega/2)$

$$\int_{-\infty}^{\infty} \frac{\sin^4 t}{t^2}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi^2 \left[\Delta(\omega/2)\right]^2 d\omega$$

$$= \frac{1}{\pi} \int_0^2 \left(1 - \frac{\omega}{2}\right)^2 d\omega$$

$$= \frac{2\pi}{3}$$

4.13 $x(t) = \exp[-\epsilon t]\, u(t)$

(a) $X(\omega) = \displaystyle\int_0^{\infty} \exp[-\epsilon t] \exp[-j\omega t]\, dt = \frac{1}{\epsilon + j\omega}$

(b) $X(\omega) = \dfrac{\epsilon - j\omega}{\epsilon^2 + \omega^2} = \dfrac{\epsilon}{\epsilon^2 + \omega^2} - j\dfrac{\omega}{\epsilon^2 + \omega^2}$

(c) $\displaystyle\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2 + \omega^2} = \begin{cases} 0 & \omega \neq 0 \\ \infty & \omega = 0 \end{cases}$ and $\displaystyle\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + \omega^2}\, d\omega = 1$

so $\displaystyle\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2 + \omega^2}$ is an impulse in ω

$$F\left\{ \lim_{\epsilon \to 0} \exp[-\epsilon t]\, u(t) \right\} = \lim_{\epsilon \to 0} \left(\frac{\epsilon}{\epsilon^2 + \omega^2}\right) - \lim_{\epsilon \to 0} j\frac{\omega}{\omega^2 + \epsilon^2}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

4.14 $H(\omega) = \mathrm{rect}\left(\dfrac{\omega}{4}\right)$; $X(\omega) = \dfrac{1}{\alpha + j\omega}$

(a) $Y(\omega) = H(\omega) X(\omega) = \begin{cases} \dfrac{1}{\alpha + j\omega} & -2 \leq \omega \leq 2 \\ 0 & \text{elsewhere} \end{cases}$

(b) Input signal energy $= \dfrac{1}{2\alpha}$

Output signal energy $= \displaystyle\frac{1}{2\pi} \int_{-2}^{2} \frac{1}{\alpha^2 + \omega^2}\, d\omega$

$$= \frac{1}{\pi} \int_0^2 \frac{1}{\alpha^2 + \omega^2}\, d\omega = \frac{1}{\pi\alpha} \tan^{-1} \frac{2}{\alpha}$$

$$\therefore \frac{1}{\pi\alpha}\tan^{-1}\frac{2}{\alpha} = \frac{1}{4\alpha} \quad \text{or} \quad \alpha = 2.$$

4.15 (a) $\Im\{x(-2t+1)\} = \frac{1}{2} e^{-j\frac{\omega}{2}} X\left(-\frac{\omega}{2}\right)$

$$= \frac{1}{2} e^{-j\frac{\omega}{2}} \cdot \frac{\frac{\omega^2}{4} - j2\omega + 2}{-\frac{\omega^2}{4} - j2\omega + 3}$$

(b) $\Im\{x(t)e^{-jt}\} = X(\omega+1) = \dfrac{(\omega+1)^2 + j4(\omega+1) + 2}{-(\omega+1)^2 + j4(\omega+1) + 3}$

(c) $\Im\left\{\dfrac{dx(t)}{dt}\right\} = j\omega X(\omega) = j\omega \dfrac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$

(d) $\Im\{x(t)\sin\pi t\} = \frac{\pi}{j}\left[X(\omega-\pi) - X(\omega+\pi)\right]$

$$= \frac{\pi}{j}\left\{\frac{(\omega-\pi)^2 + j4(\omega-\pi) + 2}{-(\omega-\pi)^2 + j4(\omega-\pi) + 3} - \frac{(\omega+\pi)^2 + j4(\omega+\pi) + 2}{-(\omega+\pi)^2 + j4(\omega+\pi) + 3}\right\}$$

(e) $\Im\{x(t) * \delta(t-1)\} = e^{-j\omega}X(\omega)$

$$= e^{-j\omega}\frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

(f) $\Im\{x(t) * x(t-1)\} = e^{-j\omega}\left[X(\omega)\right]^2$

$$= e^{-j\omega}\left[\frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}\right]^2$$

4.16 (a) $\mathbb{F}\left\{\dfrac{\sin\alpha t}{\pi t}\right\} = \text{rect}(\omega/2\alpha)$

$\mathbb{F}\left\{x(t) * \dfrac{\sin\alpha t}{\pi t}\right\} = X(\omega)\cdot\text{rect}(\omega/2\alpha)$

$= X(\omega), \quad \alpha > \omega_c$

hence $x(t) * \dfrac{\sin\alpha t}{\pi t} = x(t), \quad \alpha > \omega_c$

(b) $x(t) = \dfrac{\sin t}{t} \Rightarrow X(\omega) = \text{rect}(\omega/2), \quad \omega_c = 1$

$$\int_{-\infty}^{\infty} \frac{\sin \alpha \tau}{\tau} \frac{\sin (t-\tau)}{\pi (t-\tau)} d\tau = \begin{cases} x(t) , & \alpha \geq 1 \\ \mathcal{F}^{-1}\{ \text{rect}(\omega/2)\, \text{rect}(\omega/2\alpha) \} & \alpha \leq 1 \end{cases}$$

$$= \begin{cases} \frac{\sin t}{t} , & \alpha \geq 1 \\ \frac{\sin \alpha t}{t} , & |\alpha| \leq 1 \end{cases}$$

4.17 For an inductor, $V_L(t) = L \frac{di_L'(t)}{dt}$

$\Rightarrow \quad V_L(\omega) = L(j\omega)\, I_L(\omega)$

the inductance $= \frac{V_L(\omega)}{I_L(\omega)} = j\omega L$

For a capacitor $i_c(t) = c \frac{dv_c(t)}{dt}$

$I_c(\omega) = j\omega_c V_c(\omega)$

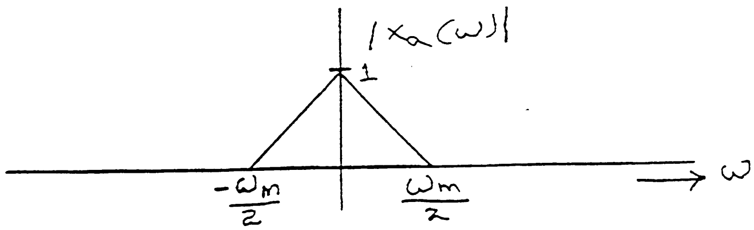
the capacitance $= \frac{V_c(\omega)}{I_c(\omega)} = \frac{1}{j\omega c}$

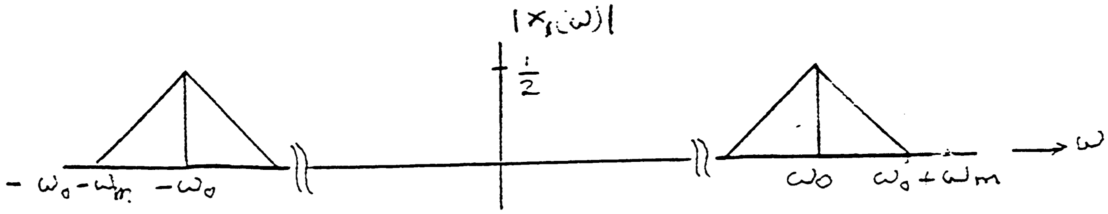
4.18 (a) $H(\omega) = \frac{1/jc\omega}{R + \frac{1}{jc\omega}} = \frac{1}{jRc\omega + 1} = \frac{1}{1 + j10\omega}$

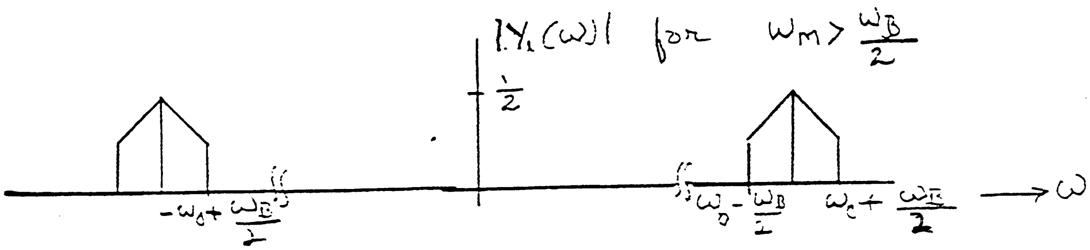
$H(\omega)$ is a low-pass filter.

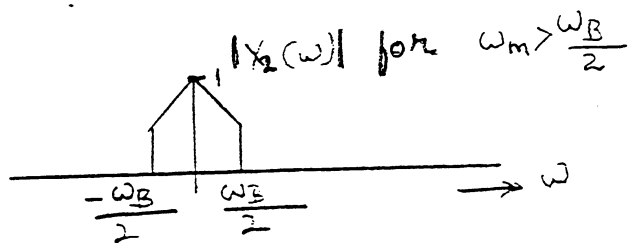
(b) $H(\omega) = \frac{R}{R + \frac{1}{jC\omega}} = \frac{j10\omega}{1 + j10\omega}$

4.19 Let $y_1(t)$ and $y_2(t)$ be the inputs to the systems with impulse responses $h_1(t)$ and $h_2(t)$. The figures below show the spectra of the various signals in the system, where $y_1(t)$ is the output of the









Since $H_2(\omega) = \text{rect}\left(\frac{\omega}{\omega_B}\right)$, $Y(\omega) = X_2(\omega)$.

When $\omega_m < \frac{\omega_B}{2}$, $Y_1(\omega) = X_1(\omega)$ and $Y(\omega) = X_a(\omega)$

4.20 $\quad x(t) = Sa\left(\frac{3\omega_B t}{2}\right) \longleftrightarrow X(\omega) = \frac{2\pi}{3\omega_B}\,\text{rect}\left(\omega/3\omega_B\right)$

$$y(t) = \frac{\pi}{3\omega_B}\,Sa\left(\frac{\omega_B t}{2}\right)$$

4.21 (a) $H(\omega) = \int_{-\infty}^{\infty} \frac{e^{-j\omega t}}{\pi t}\, dt$

$$= \int_{-\infty}^{0} \frac{e^{-j\omega t}}{\pi t}\, dt + \int_{0}^{\infty} \frac{e^{-j\omega t}}{\pi t}\, dt$$

$$= \int_{0}^{\infty} \frac{e^{-j\omega t} - e^{j\omega t}}{\pi t}\, dt = \int_{0}^{\infty} -2j\, \frac{\sin \omega t}{\pi t}\, dt$$

$$= -\frac{2j}{\pi} \times \frac{\pi}{2} \operatorname{sgn} \omega = -j \operatorname{sgn} \omega$$

(b) $X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

$Y(\omega) = \mathfrak{F}\{\text{Hilbert transform of } x(t)\}$

$$= (-j \operatorname{sgn} \omega)\, \pi\, [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$= \frac{\pi}{j} (\delta(\omega + \omega_0) - \delta(-\omega_0))$$

$\therefore y(t) = \sin \omega_0 t$

4.22 $\mathbb{F}\{R_x(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, x(\tau + t) \exp[-j\omega t]\, dt\, d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} x(\tau + t) \exp[-j\omega t]\, dt\, d\tau$$

$$= X(\omega) \int_{-\infty}^{\infty} x(\tau) \exp[j\omega\tau]\, d\tau$$

$$= |X(\omega)|^2, \quad \text{for real-valued } x(t)$$

4.23 $R_Y(t) = \int_{-\infty}^{\infty} y(\tau)\, y(\tau+t)\, d\tau$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\, x(\tau-u)\, h(v)\, x(\tau+t-v)\, du\, dv\, d\tau$$

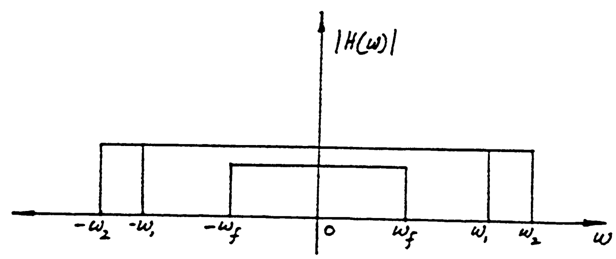
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\, h(v)\, R_x(t-v+u)\, du\, dv$$

$$= R_x(t) * h(t) * h(-t)$$

4.24 $x(t) = W_1\, Sa(\omega_1 t) + \omega_2\, Sa(\omega_2 t) \qquad \omega_1 < \omega_2$

$\longleftrightarrow \; X(\omega) = \pi\, \mathrm{rect}(\omega/2\omega_1) + \pi\, \mathrm{rect}(\omega/2\omega_2)$

(a) $0 < W_f < W_1 < W_2$



$y(t) = F^{-1}[H(\omega)\, X(\omega)]$

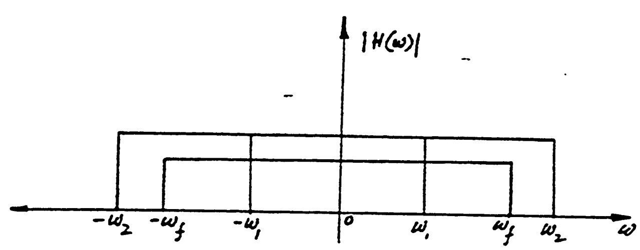
$= F^{-1}[\pi\, \mathrm{rect}(\omega/2\omega_1) + \pi\, \mathrm{rect}(\omega/2\omega_2)] \qquad -W_f \le \omega \le W_f$

$= F^{-1}[2\pi\, \mathrm{rect}(\omega/2\omega_f)]$

$= 2W_f\, Sa(W_f t)$

$= 2\,\dfrac{\sin W_f t}{t}$

(b) $\omega_1 < W_f < \omega_2$



$y(t) = F^{-1}[H(\omega)\, X(\omega)]$

$= F^{-1}\left[\mathrm{rect}(\omega/2\omega_f)\; \pi\, \mathrm{rect}(\omega/2\omega_1) + \mathrm{rect}(\omega/2\omega_f) \cdot \pi\, \mathrm{rect}\left(\frac{\omega}{2\omega_2}\right)\right]$

$= F^{-1}[\pi\, \mathrm{rect}(\omega/2\omega_1) + \pi\, \mathrm{rect}(\omega/2\omega_f)]$

$= W_1 Sa(W_1 t) + W_f Sa(W_f t)$

$= \dfrac{\sin \omega_1 t}{t} + \dfrac{\sin W_f t}{t}$

(c) $W_1 < W_2 < W_f$

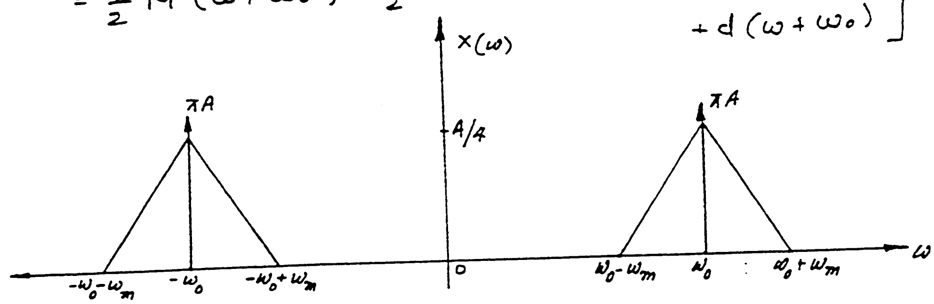
from (a), (b) we conclude that

$y(t) = \dfrac{\sin W_1 t}{t} + \dfrac{\sin W_2 t}{t}$

4.25 $x(t) = (m(t) + 4) \cos \omega_0 t$

(a) $X(\omega) = F[m(t) \cos \omega_0 t + A \cos \omega_0 t]$

$= \frac{1}{2} M(\omega + \omega_0) + \frac{1}{2} M(\omega - \omega_0) + A\pi [\delta(\omega - \omega_0) + d(\omega + \omega_0)]$

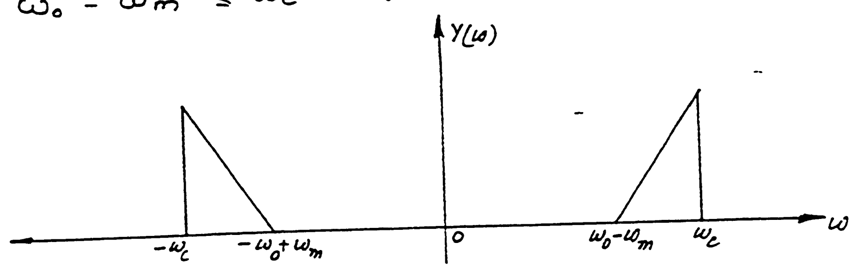


(b) $y(t) = h_c(t) * x(t) \implies Y(\omega) = H_c(\omega) X(\omega)$

(i) $0 \le \omega_c \le \omega_0 - \omega_m$

$y(t) = 0$

(ii) $\omega_0 - \omega_m \le \omega_c < \omega_0$



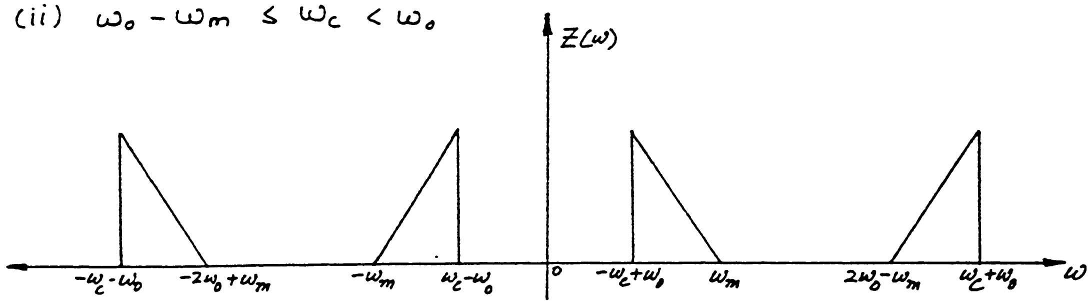
(iii) $\omega_c > \omega_0 + \omega_m$

$Y(\omega) = X(\omega)$, see part (a)

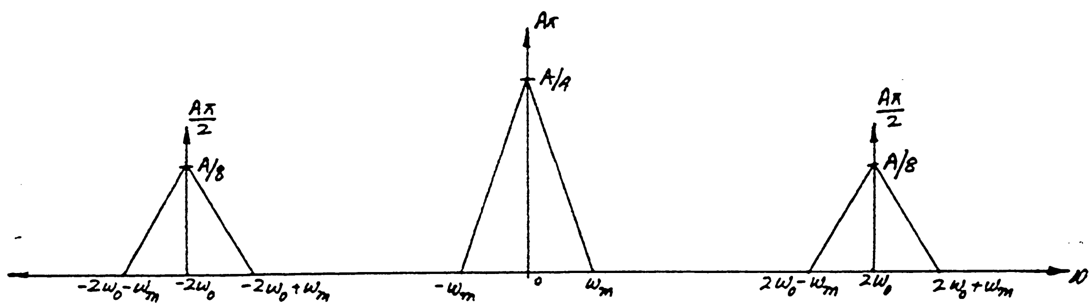
(c) $z(t) = y(t)\cos\omega_0 t \implies Z(\omega) = \frac{1}{2}Y(\omega-\omega_0) + \frac{1}{2}Y(\omega+\omega_0)$

(i) $0 \le \omega_c < \omega_0 - \omega_m$, $Y(\omega) = 0 \implies Z(\omega) = 0$

(ii) $\omega_0 - \omega_m \le \omega_c < \omega_0$

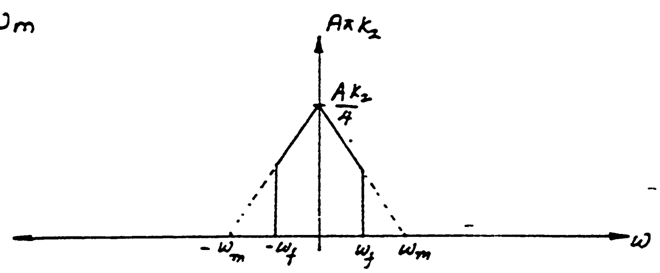


(iii) $\omega_c > \omega_0 + \omega_m$

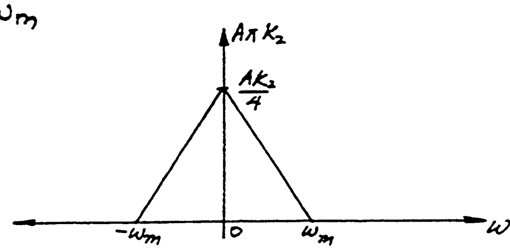


(d) $V(t) = h_f(t) * z(t) \implies V(\omega) = H_f(\omega)Z(\omega)$, $\omega_c > \omega_0 + \omega_m$

(i) $\omega_f < \omega_m$



(ii) $\omega_f < 2\omega_0 - \omega_m$



(iii) For $\omega_f > 2\omega_0 + \omega_m$, $V(\omega) = Z(\omega)$ in (c)-(ii) and the spectrum is the same as in (c)-(ii)

4.26 $z(t) = [x(t) A\cos\omega_0 t][A\cos(\omega_0 t + \theta)]$

$$= \frac{A^2}{2} x(t)[\cos\theta + \cos(2\omega_0 t + \theta)]$$

If we pass $z(t)$ through a low-pass filter with cutoff ω_m

$$\hat{x}(t) = \frac{A^2}{2}\cos\theta \; x(t)$$

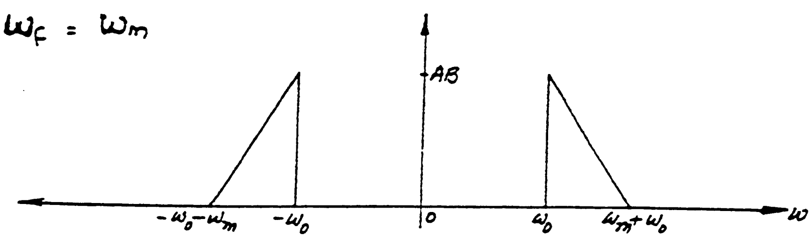
Desired output is $x_d(t) = x(t)$

$$\therefore \hat{x}(t) = \frac{A^2}{2}\cos\theta \; x_d(t)$$

4.27 (a) $x(t) = m(t)\cos\omega_0 t \Rightarrow X(\omega) = \frac{1}{2}M(\omega+\omega_0) + \frac{1}{2}M(\omega-\omega_0)$

$y(t) = h_f(t) * x(t) \Rightarrow Y(\omega) = H_f(\omega)\,X(\omega)$

for $\omega_f = \omega_m$



(b) $$h_f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_f(\omega)\cdot\exp[j\omega t]\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0+\omega_m}^{-\omega_0} B\exp[j\omega t]\,d\omega + \frac{1}{2\pi}\int_{\omega_0}^{\omega_m+\omega_0} B\exp[j\omega t]\,d\omega$$

$$= \frac{B}{2\pi}\int_{\omega_0}^{\omega_0+\omega_m}\exp[-j\omega t]\,d\omega + \frac{B}{2\pi}\int_{\omega_0}^{\omega_0+\omega_m}\exp[j\omega t]\,d\omega$$

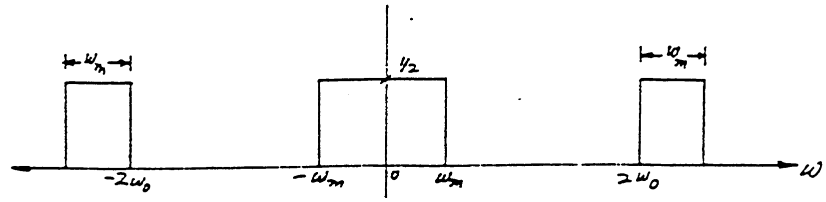
$$= \frac{B}{\pi t}\left[\sin(\omega_0+\omega_m)t - \sin\omega_0 t\right]$$

$h_f(t) \neq 0$ for $t < 0$

implies the filter is an unrealizable

4.28 $y(t) = [h_1(t) * m(t)]\cos\omega_0 t + [h_2(t) * m(t)]\sin\omega_0 t$

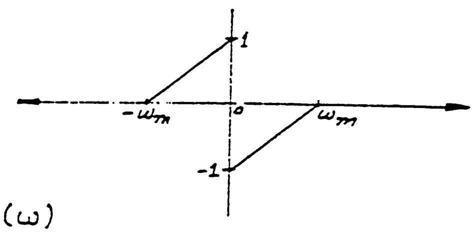
$H_1(\omega) =$



$$H_2(\omega)M(\omega) = \frac{-1}{2j}\left[H_0(\omega-\omega_0) - H_0(\omega+\omega_0)\right]M(\omega)$$

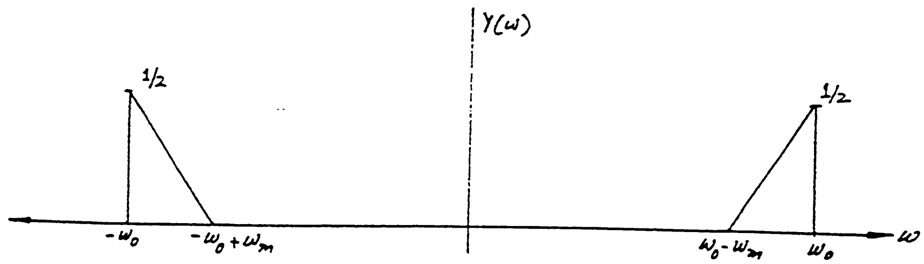
$$= \frac{-1}{2j}\left[H_0(\omega-\omega_0)M(\omega) - H_0(\omega+\omega_0)M(\omega)\right]$$

where $H_0(\omega-\omega_0)M(\omega) - H_0(\omega+\omega_0)M(\omega) = F(\omega)$



$H_1(\omega)M(\omega) = \frac{1}{2}M(\omega)$

$$Y(\omega) = \frac{1}{4}\left[M(\omega-\omega_0) + M(\omega+\omega_0)\right] + \frac{1}{4}\left[F(\omega-\omega_0) - F(\omega+\omega_0)\right]$$



(b) $Y(\omega) = H_1(\omega)M(\omega) * \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + \left[H_2(\omega)M(\omega)\right]$

$$* -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$= \frac{\pi}{2\pi}\left\{H_1(\omega-\omega_0)M(\omega-\omega_0) + H_1(\omega+\omega_0)M(\omega+\omega_0)\right]$$

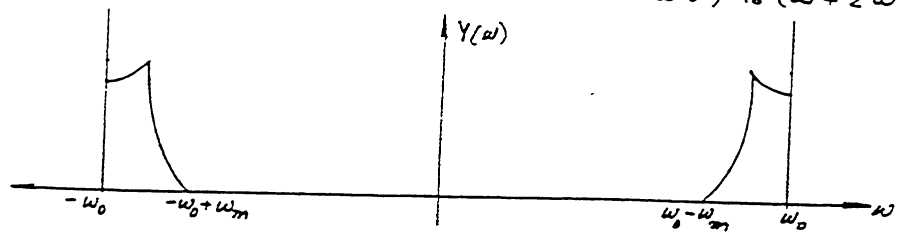
$$-j\frac{\pi}{2\pi}\left[H_2(\omega-\omega_0)M(\omega-\omega_0) - H_2(\omega+\omega_0)M(\omega+\omega_0)\right]$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[H_0(\omega-2\omega_0) + H_0(\omega)\right]M(\omega-\omega_0)\right] + \frac{1}{2}\left[(H_0(\omega) + H_0(\omega+2\omega_0))\right.$$

$$\left.M(\omega+\omega_0)\right] - \frac{j}{2}\left\{-\frac{1}{2j}\left[(H_0(\omega-2\omega_0) - H_0(\omega))M(\omega-\omega_0)\right.\right.$$

$$-\frac{1}{\ }\left[H_0(\omega) - H_0(\omega+2\omega_0)\right]M(\omega+\omega_0)\Big]$$

$$Y(\omega) = \frac{1}{2}\left[M(\omega - \omega_o)H_o(\omega - 2\omega_o) + M(\omega + \omega_o)H_o(\omega + 2\omega_o)\right]$$



We obtain a distorted LSB signal. The conclusion is: as long as $H_o(\omega - \omega_o) + H_o(\omega + \omega_o)$ = constant in the frequency band of the signal, we can obtain undistorted LSB.

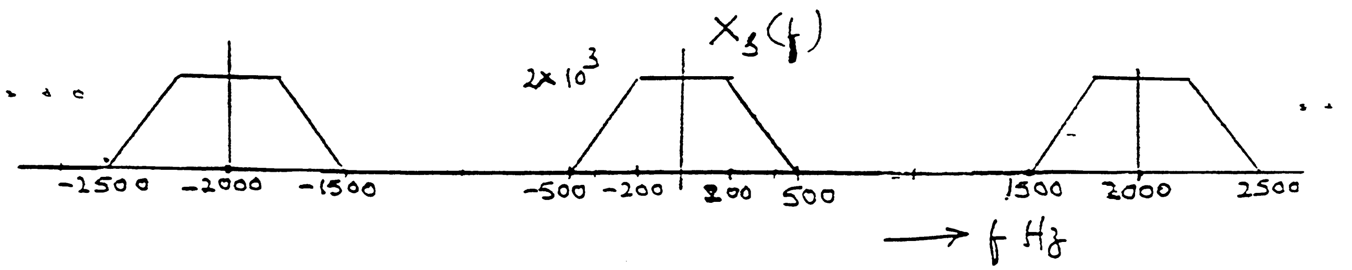
4.29 Bandwidth of $z(t)$ is 500 Hz

(a) $T_{max} = \frac{1}{2 \times 500} = 1$ ms

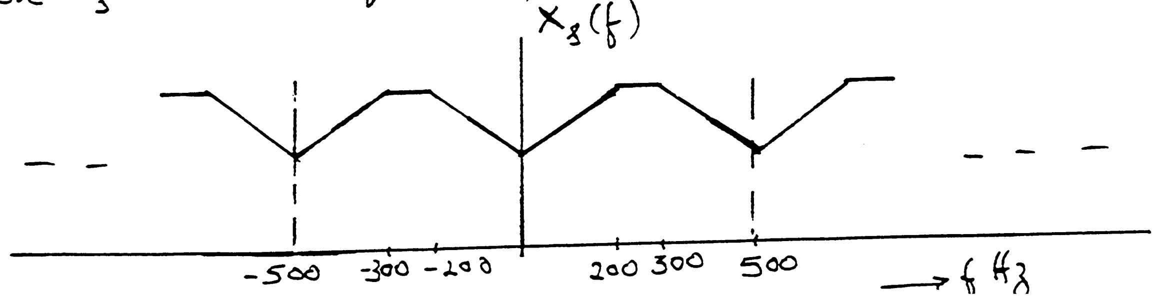
(b) $X(\omega) = \text{rect}\left(\frac{\omega}{600\pi}\right)$, $Y(\omega) = \text{rect}\left(\frac{\omega}{1400\pi}\right)$

$$\therefore Z(\omega) = X(\omega) * Y(\omega) = \begin{cases} 1000\pi + \omega & -1000\pi < \omega < -400\pi \\ 600\pi & -400\pi < \omega < 400\pi \\ 1000\pi - \omega & 400\pi < \omega < 1000\pi \\ 0 & \text{otherwise} \end{cases}$$

For $T_s = 0.5$ msecs, $f_s = \frac{1}{T_s} = 2000$ Hz



For $T_s = .2$ msec, $f_s = 500$ Hz



4.30 (a) The pulse train $p(t)$ has the Fourier series expansion

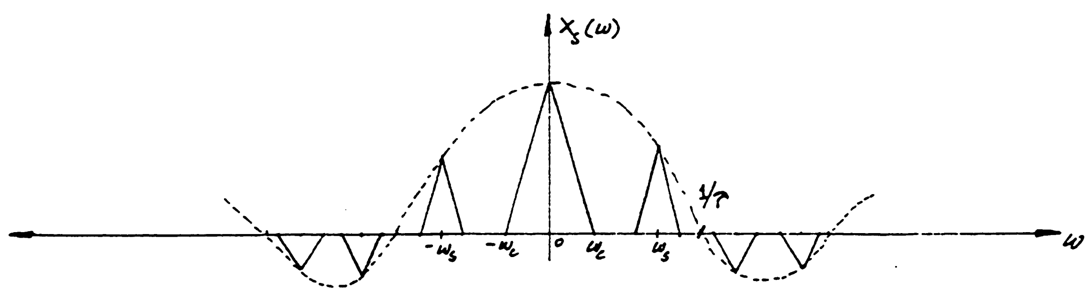
$$p(t) = \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/2)}{k\omega_s \tau/2} \exp[jk\omega_s t], \quad \omega_s = \frac{2\pi}{T}$$

Its spectrum

$$P(\omega) = \frac{2\pi\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/2)}{k\omega_s \tau/2}\, \delta(\omega - k\omega_s)$$

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$= \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/k)}{k\omega_s \tau/2}\, X(\omega - k\omega_s)$$



(b) Yes, so long as $\omega_s \geq 2\omega_c$. By applying $x_s(t)$ to a low-pass filter of bandwidth ω_c, $x(t)$ is easily recovered without distortion.

4.31 (a) Assuming ideal rectangular pulses, the flat-top sampled signal is

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(tT)\, \mathrm{rect}((t-kT)/\tau)$$

(b) $$X_s(\omega) = \frac{\tau}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \frac{\sin \omega\tau/2}{\omega\tau/2}, \quad \omega_s = \frac{2\pi}{T}$$

(c) $X_s(\omega)$ appears on the surface to be similar to part (a) of problem 4.30 There is an important difference, however A low-pass filter operating on $X_s(\omega)$ in the flat-top sampled signal will not give distortion-free output proportional to $x(t)$.

(d) Using a low-pass filter alone, the output spectrum would be $\dfrac{\tau}{T_s}\,\dfrac{X(\omega)\sin(\omega\tau/2)}{\omega\tau/2}$,

which is clearly not proportional to $X(\omega)$ as needed Therefore, $x(t)$ cannot be recovered distortion-free

(e) The factor $Q(\omega) = \sin(\omega\tau/2)/(\omega\tau/2)$ represents distortion which may be corrected by adding a second filter, called an equalizing filter.

It must have a transfer function.

$$H_{eq}(\omega) = 1/Q(\omega) \qquad |\omega| < \omega_c$$

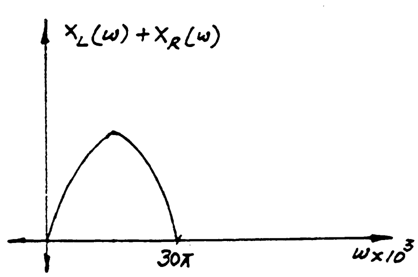
$$= \frac{\omega\tau/2}{\sin(\omega\tau/2)} \qquad |\omega| < \omega_c$$

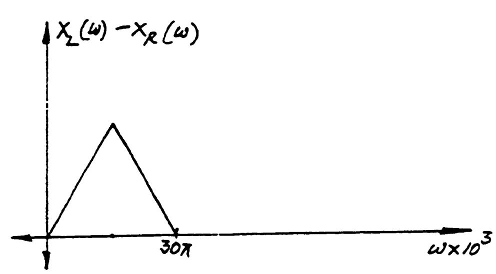
$$= \text{arbitrary} \qquad \text{elsewhere}$$

The equalized output is $X_o(\omega) = \dfrac{\tau}{T}X(\omega)$

and $x_o(t) = \dfrac{\tau}{T}x(t)$

4.32

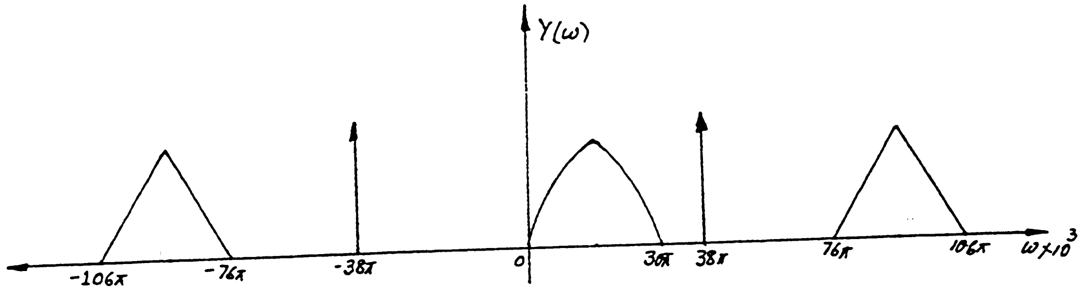




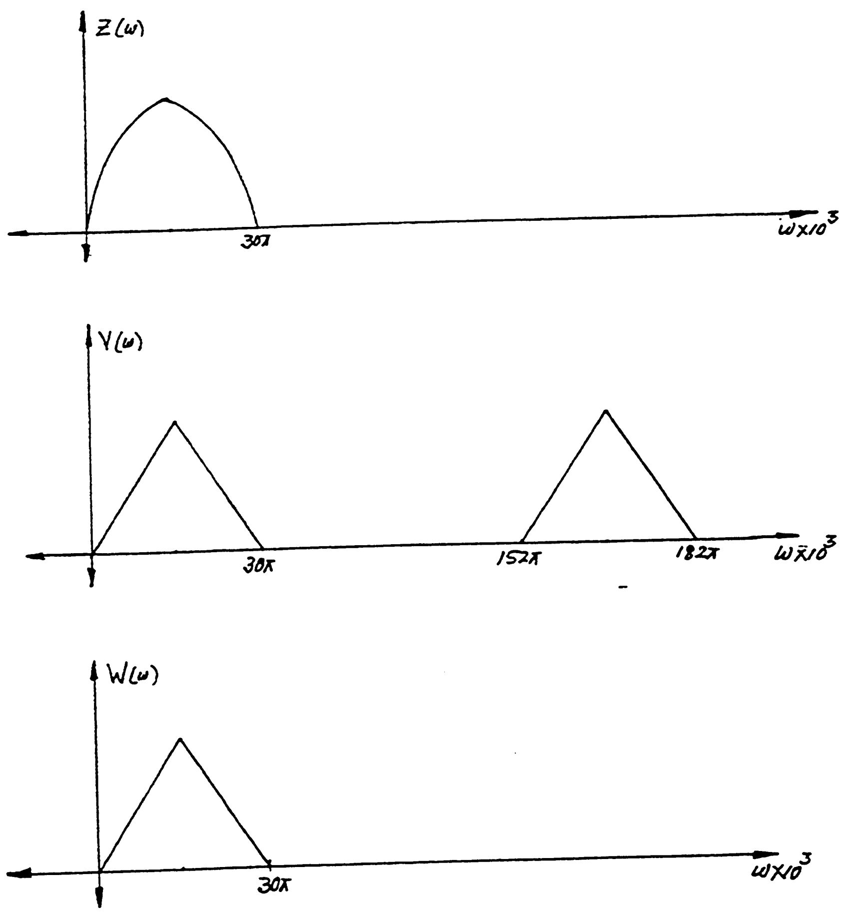
(a) $y(t) = \left(x_L(t) - x_R(t) \cos 2\omega_1 t + x_L(t) + x_R(t) + \cos \omega_1 t \right)$

$$Y(\omega) = \frac{1}{2}\left[X_L(\omega - 2\omega_1) - X_R(\omega - 2\omega_1) + X_L(\omega + 2\omega_1) - X_R(\omega + 2\omega_1) \right]$$

$$+ X_L(\omega) + X_R(\omega) + \frac{1}{2}\delta(\omega - \omega_1) + \frac{1}{2}\delta(\omega + \omega_1)$$



(b)



(c) $W(\omega) = X_L(\omega) - X_R(\omega)$

$Z(\omega) = X_L(\omega) + X_R(\omega)$

Then

$X_L(\omega) = \left(W(\omega) + Z(\omega)\right)/2$

and $X_R(\omega) = \left(Z(\omega) - W(\omega)\right)/2$

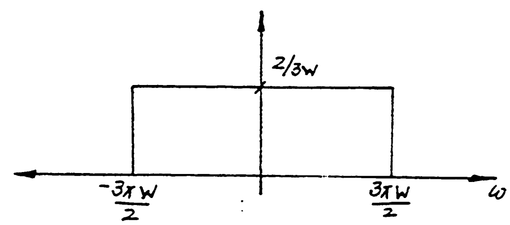
4.33 $\displaystyle\sum_{n=-\infty}^{\infty} \delta'(t-nT) = \sum_{n=-\infty}^{\infty} \frac{d}{dt}\delta(t-nT)$

but $\displaystyle F\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T}\right) = \omega_0 \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$

$\displaystyle\Rightarrow \sum_{n\to\infty}^{\infty}\delta'(t-nT) \longleftrightarrow \sum_{n=-\infty}^{\infty} jn\omega_0^2\delta(\omega-n\omega_0) = j\omega_0^2\sum_{n=-\infty}^{\infty} n\delta(\omega-n\omega_0)$

4.34 (a) $\text{Sinc}(3Wt/2)$

Bandwidth $= \dfrac{3\pi W}{2}$



(b) $\exp(-3t)u(t) = x(t)$

$|X(\omega)| = \dfrac{1}{\sqrt{9^2+\omega^2}}$

let $\dfrac{|X(\omega)|^2}{|X(0)|^2} = \dfrac{1}{2} \Longrightarrow \omega = \pm 3$

$\therefore$ Bandwidth $= 3$

(c) $\displaystyle\int_{-W_B}^{W_B}|X(\omega)|^2 d\omega = \int_{-W_B}^{W_B}\frac{1}{9+\omega^2}d\omega = 0.95\int_{-\infty}^{\infty}\frac{1}{9+\omega^2}d\omega = 0.95\times\frac{\pi}{3}$

$\Longrightarrow \dfrac{2}{3}\tan^{-1}\left(\dfrac{W_B}{3}\right) = 0.95\times\dfrac{\pi}{3}$

$W_B = 3\tan\left(\dfrac{0.95\times\pi}{2}\right)$

$\simeq 38$ rad/sec

(d) $x(t) = \sqrt{\frac{\alpha}{\pi}} \exp[-\alpha t^2] \longleftrightarrow X(\omega) = \exp[-\omega^2/4\alpha]$

$$\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \omega^2 \exp[-\omega^2/2\alpha] d\omega = 2\int_0^{\infty} \omega^2 \exp[-\omega^2/2\alpha] d\omega$$

$$= \alpha\sqrt{2\alpha\pi}$$

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\int_0^{\infty} \exp[-\omega^2/2\alpha] d\omega = \sqrt{2\alpha\pi}$$

$$B^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega} = \frac{\alpha\sqrt{2\alpha\pi}}{\sqrt{2\alpha\pi}} = \alpha$$

$$B_{rms} = \sqrt{\alpha}$$

4.35(a) $C_n = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} X_s(\omega) \exp[-j n \frac{2\pi}{\omega_s} \omega] d\omega, \quad \omega_s = \frac{2\pi}{T}$

Since in the central period $X_s(\omega) = X(\omega)$

$$= \frac{1}{\omega_s} \int_{-\omega_B}^{\omega_B} X(\omega) \exp[-jn \frac{2\pi}{\omega_s} \omega] d\omega$$

$$= \frac{2\pi}{\omega_s} x(-\frac{n 2\pi}{\omega_s}) = \frac{2\pi}{\omega_s} x(-nT)$$

(b) $X_s(\omega) = \sum_{n=-\infty}^{\infty} C_n \exp[jn \frac{2\pi}{\omega_s} \omega]$

$$= \sum_{n=-\infty}^{\infty} \frac{2\pi}{\omega_s} x(-nT) \exp[jn \frac{2\pi}{\omega_s} \omega]$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{\omega_s} x(kT) \exp[-jk \frac{2\pi}{\omega_s} \omega]$$

$$x_s(t) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{\omega_s} x(kT) \delta(t - k\frac{2\pi}{\omega_s})$$

(c) $X(\omega) = X_s(\omega) \operatorname{rect}(\omega/2\omega_B)$

taking the inverse Fourier transform yields

$$x(t) = x_s(t) * \frac{\sin \omega_B t}{\pi t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\omega_B}{\omega_s} x(kT) \frac{\sin \omega_B (t-kT)}{\omega_B (t-nT)}$$

4.35(a) $$x(t) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp[-t^2/2\Delta^2] \longleftrightarrow X(\omega) = \exp[-\Delta^2\omega^2/2]$$

$$T = 2\sqrt{\frac{\Delta^2}{4\sqrt{2\pi\Delta^2}} \Big/ \frac{1}{2\sqrt{2\pi\Delta^2}}} = \sqrt{2}\,\Delta$$

$$X_m = 1 \qquad B_{eq} = \frac{\int_0^\infty \exp[-\Delta^2\omega^2]\,d\omega}{|X(0)|^2} = \frac{\sqrt{2\pi}}{2\Delta}$$

$$\therefore B_{eq} T = \sqrt{\pi}$$

(b) $$x(t) = \frac{\sin 2\pi W t}{\pi t} \longleftrightarrow X(\omega) = \mathrm{rect}(\omega/4\pi W)$$

$$T = \frac{1}{W} \qquad B = 2\pi W$$

$$\Longrightarrow BT = 2\pi$$

(c) $$x(t) = Ae^{-\alpha t}u(t) \longleftrightarrow X(\omega) = \frac{A}{\alpha + j\omega}$$

$$T = \frac{1}{\alpha} \qquad B = \alpha$$

$$\therefore BT = 1$$

CHAPTER 5

5.1 (a) $x(t) = \exp[t+1]$ does not have a bilateral Laplace transform because $\int_{-\infty}^{\infty} x(t)\exp[-st]\,dt$ does not converge

(b) $$X_B(s) = \int_{-\infty}^{\infty} \exp[bt]u(-t)\exp[-st]\,dt$$

$$= \frac{1}{b-s}, \quad Re\{s\} < b$$

(c) $x(t) = |t|$ does not have a bilateral Laplace transform

(d) $x(t) = (1-|t|)$ does not have a bilateral Laplace transform.

(e) $$X_B(s) = \int_{-\infty}^{\infty} \exp[-2|t|]\exp[-st]\,dt$$

$$= \frac{4}{4-s^2} \qquad -2 < Re\{s\} < 2$$

(f) $x(t) = t^2\exp[-t]u[-t]$ does not have a bilateral Laplace transform

(g) $$X_B(s) = \int_{-\infty}^{\infty} \cos at\, u(-t)\exp[-st]\,dt$$

$$= \frac{-s}{s^2+a^2}, \quad Re\{s\} < 0$$

(h) $$X_B(s) = \int_{-\infty}^{\infty} \sinh at\, u(-t)\exp[-st]\,dt$$

$$= \frac{2a}{a^2-s^2}, \quad Re\{s\} < -a$$

5.2

(i) $$X_1(s) = \int_0^2 t\, e^{-st}\,dt = \frac{1-e^{-2s}}{s^2} - \frac{e^{-2s}}{s}$$

(ii) $X_2(s) = X_1(s) + \frac{1}{2} = \frac{1-e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + \frac{1}{2}$

(iii) $X(s) = \int_{-1}^{1} e^{-st}\,dt = \frac{e^{-s} - e^{s}}{s}$

5.3 (a) Does not have a bilateral Laplace transform (ROC for $x_+(t)$ is $Re\{s\} > 1$ and the ROC for $x_-(t)$ is $Re\{s\} < 1$ and the overlap is the null set)

(b) $X(s) = \int_0^\infty \exp[bt]\exp[-st]dt = \frac{1}{s-b}$

$X_B(s) = \mathcal{L}\{\exp[-bt]\}_{s\to -s} = \left\{\frac{1}{s+b}\right\}_{s=-s} = \frac{1}{-s+b}$, $Re\{s\} < b$

(c) Does not have a bilateral Laplace transform

(d) Does not have a bilateral Laplace transform

(e) $X_+(s) = \int_0^\infty \exp[-2t]\exp[-st]dt = \int_0^\infty \exp[-(2+s)t]dt = \frac{1}{s+2}$

noncausal part $= \mathcal{L}\{\exp[-2t]\}_{s\to -s} = \frac{1}{-s+2}$

$X(s) = \frac{1}{s+2} + \frac{1}{-s+2} = \frac{4}{4-s^2}$

(f) Does not have a bilateral Laplace transform.

(g) $X_+(s) = 0$

$X(s) = \mathcal{L}\{\cos -at\}\Big|_{s=-s} = \frac{-s}{s^2+a^2}$

(h) $X_+(s) = 0$

$X(s) = \mathcal{L}\{\sinh(-at)\}_{s=-s} = \frac{2a}{a^2-s^2}$

5.4 (a) $Y(s) = 3 \cdot \frac{1}{3} X\left(\frac{s}{3}\right) = \dfrac{\left(\frac{s}{3}\right)^3 + 2\left(\frac{s}{3}\right)^2 + 3\left(\frac{s}{3}\right) + 1}{\left(\frac{s}{3}\right)^4 + 2\left(\frac{s}{3}\right)^3 + 2\left(\frac{s}{3}\right)^2 + 2}$

$$= \frac{3(s^3 + 6s^2 + 27s + 54)}{s^4 + 6s^3 + 18s^2 + 54s + 162}$$

(b) $Y(s) = -\dfrac{dX(s)}{ds} = -\dfrac{s^6 + 4s^5 + 11s^4 + 16s^3 + 8s^2 - 2}{(s^4 + 2s^3 + 2s^2 + 2s + 2)^2}$

(c) $y(t) = (t-1)x(t-1) + x(t-1)$

$\therefore Y(s) = (e^{-s} + 1)X(s) = \dfrac{(e^{-s}+1)(s^3 + 2s^2 + 3s + 2)}{s^4 + 2s^3 + 2s^2 + 2s + 2}$

(d) $Y(s) = sX(s) = \dfrac{s^4 + 2s^3 + 3s^2 + 2s}{s^4 + 2s^3 + 2s^2 + 2s + 2}$

(e) $Y(s) = e^{-s}X(s) + sX(s)$

$$= \frac{(e^{-s} + s)(s^3 + 2s^2 + 3s + 2)}{s^4 + 2s^3 + 2s^2 + 2s + 2}$$

(f) $Y(s) = \dfrac{1}{s}X(s) = \dfrac{s^3 + 2s^2 + 3s + 2}{s[s^4 + 2s^3 + 2s^2 + 2s + 2]}$

5.5 $\mathcal{L}\{t^n\} = \displaystyle\int_0^\infty t^n \exp[-st]\,dt$

$$= \frac{t^n \exp[-st]}{-s}\Bigg|_0^\infty - \frac{n}{(-s)}\int_0^\infty t^{n-1}\exp[-st]\,dt$$

$$\therefore = \frac{n}{s}\int_0^\infty t^{n-1}\exp[-st]\,dt$$

$$= \frac{n}{s}\left[\frac{t^{n-1}\exp[-st]}{-s}\Bigg|_0^\infty - \frac{n-1}{(-s)}\int_0^\infty t^{n-1}\exp[-st]\,dt\right]$$

$$= \frac{n(n+1)}{s^2}\int_0^\infty t^{n-2}\exp[-st]\,dt$$

$$= \frac{n(n-1)}{s^2}\left[\frac{t^{n-2}\exp[-st]}{-s}\Bigg|_0^\infty - \frac{n-2}{(-s)}\int_0^\infty t^{n-3}\exp[-st]\,dt\right]$$

$$= \frac{n(n-1)(n-2)}{s^3}\int_0^\infty t^{n-3}\exp[-st]\,dt = \cdots$$

$$= \frac{n(n-1)(n-2)\cdots 1}{s^n}\int_0^\infty \exp[-st]\,dt = \frac{n!}{s^{n+1}}$$

5.6 $\mathcal{L}\{t^a\} = \int_0^\infty t^a \exp[-st]\,dt$

$$\Longrightarrow \mathcal{L}\{t^a\} = \int_0^\infty t^a \exp[-st]\,dt = \int_0^\infty t^{(a+1-1)} \exp[-st]\,dt$$

let $st = t'$ $\quad dt = \frac{1}{s}dt'$

$$\therefore \mathcal{L}\{t^a\} = \int_0^\infty \left(\frac{t'}{s}\right)^{(a+1-1)} \exp[-t']\,\frac{1}{s}\,dt' = \int_0^\infty \frac{t'^{(a+1-1)}}{s^{a+1}} \exp[-t']\,dt'$$

Compare with Gamma function

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}} \qquad a>0$$

5.7 $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

where
$$\Gamma(n+1) = n\Gamma(n) = n\Gamma(n-1+1) = n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\cdots 1\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty \exp[-t]\,dt = 1$$

$$\therefore \Gamma(n+1) = n!$$

i.e. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

5.8 $\mathcal{L}\{\cos\omega t\} = \int_0^\infty \cos\omega t\,\exp[-st]\,dt$

let $u = \cos\omega t$ $\quad dv = \exp[-st]\,dv$

$$\int_0^\infty \cos\omega t\,\exp[-st]\,dt = \cos\omega t\left(-\frac{\exp[-st]}{s}\right)\Big|_0^\infty - \int\left(\frac{-\exp[-st]}{s}\right)(-\sin\omega t\,.\,\omega)\,dt$$

$$= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty \exp[-st]\sin\omega t\,dt$$

let $u = \sin\omega t$ $\quad dv = \exp[-st]\,dt$

$$\int_0^\infty \cos\omega t\,\exp[-st]\,dt = \frac{1}{s} - \frac{\omega}{s}\left[\sin\omega t\left(-\frac{\exp[-st]}{s}\right)\Big|_0^\infty - \int_0^\infty\left(-\frac{\exp[-st]}{s}\right)\omega\cdot\cos\omega t\cdot dt\right]$$

$$\Longrightarrow \left(1+\frac{\omega^2}{s^2}\right)\int_0^\infty \cos\omega t \exp[-st]\,dt = \frac{1}{s}$$

$$\text{or} \quad \int_0^\infty \cos\omega t \exp[-st]\,dt = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\} = \int_0^\infty \sin\omega t \exp[-st]\,dt$$

let $u = \sin\omega t \qquad dv = \exp[-st]\,dt$

$$\int_0^\infty \sin\omega t \exp[-st]\,dt = \sin\omega t\left(-\frac{\exp[-st]}{s}\right)\Bigg|_0^\infty - \int_0^\infty\left(-\frac{\exp[-st]}{s}\,\omega\cos\omega t\,dt\right)$$

$$= \frac{\omega}{s}\int_0^\infty \exp[-st]\cos\omega t\,dt$$

Since $\displaystyle\int_0^\infty \exp[-st]\cos\omega t\,dt = \frac{s}{s^2+\omega^2}$

$$\therefore \int_0^\infty \sin\omega t \exp[-st]\,dt = \frac{\omega}{s}\left[\frac{s}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

5.9 $\cosh t = \cos(jt) \quad \therefore \cosh(\omega_0 t) = \cos(j\omega_0 t)$

$\sinh t = -j\sin(jt) \quad \therefore \sinh(\omega_0 t) = -j\sin(j\omega_0 t)$

$$\therefore \mathcal{L}[\cosh t\, u(t)] = \mathcal{L}[\cos(j\omega_0 t\, u(t)]$$

$$= \frac{1}{j}\,\frac{s/j}{\left(\frac{s}{j}\right)^2+\omega_0^2} = \frac{s}{s^2-\omega_0^2}$$

$$\mathcal{L}[\sinh t\, u(t)] = -j\,\mathcal{L}[\sin(j\omega_0 t)\,u(t)]$$

$$= -j \times \frac{1}{j}\,\frac{\omega_0}{\left(\frac{s}{j}\right)^2+\omega_0^2} = \frac{\omega_0}{s^2-\omega_0^2}$$

5.10 (a) $$x(0^+) = \lim_{s\to\infty}\frac{s}{s+a} = 1$$

$$x(\infty) = \lim_{s\to 0}\frac{s}{s+a} = 0$$

(b) $x(0^+) = \lim_{s \to \infty} \dfrac{s}{(s+a)^n} = 0$

Since $x(t) = \dfrac{1}{n-1!} t^{n-1} e^{-at} u(t)$, $x(\infty)$ does

(c) $x(0^+) = \lim_{s \to \infty} \dfrac{6}{s^2+25} = 0$

Since $x(t)$ contains a sinusoidal term, $x(\infty)$ does not exist.

(d) $x(0^+) = \lim_{s \to \infty} \dfrac{s(s+2)}{s+3} = \infty$

$x(\infty) = \lim_{s \to 0} \dfrac{s(s+2)}{s+3} = 0$

[Note that $x(t) = \delta(t) - e^{-3t} u(t)$]

(e) $x(0^+) = \lim_{s \to \infty} \dfrac{s(s^2+s+3)}{s^3+4s^2+2s+2} = 1$

$x(\infty) = \lim_{s \to 0} \dfrac{s(s^2+s+3)}{s^3+4s^2+2s+2} = 0$

(f) $x(0^+) = \lim_{s \to \infty} \dfrac{s(s)}{s^2-2s-3} = 1$

$x(\infty) = \lim_{s \to 0} \dfrac{s^2}{s^2-2s-3} = 0$

5.11 (a) $X(s) = \dfrac{s+2}{(s-2)(s+1)} = \dfrac{4/3}{s-2} + \dfrac{-1/3}{s+1}$

$x(t) = \left[\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}\right] u(t)$

(b) $X(s) = \dfrac{s^2+8}{s(s^2+16)} = \dfrac{1/2}{s} + \dfrac{s}{s^2+16} - \dfrac{1}{8}\,\dfrac{4}{s^2+16}$

$x(t) = u(t) + \cos 2t\, u(t) - \frac{1}{8} \sin t\, u(t)$

(c) $X(s) = \dfrac{s}{s^2+4} + \dfrac{s+1}{s^2+2s+2}$

$x(t) = \cos 2t\, u(t) + e^{-t}\cos t\, u(t)$

(d) $X(s) = \dfrac{s^2}{s^2+3s+2} = 1 - \dfrac{4}{s+2} + \dfrac{1}{s+1}$

$x(t) = \delta(t) - 4e^{-2t}u(t) + e^{-t}u(t)$

(e) $X(s) = \dfrac{s^2-s+1}{s^3-2s^2+s} = \dfrac{1}{s} + \dfrac{1}{(s-1)^2}$

$x(t) = u(t) + t e^{t} u(t)$

(f) Let $X_1(s) = \dfrac{s+2}{s^2+2s+1} = \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2}$

Then $x_1(t) = e^{-t}u(t) + t e^{-t} u(t)$

Since $X(s) = e^{-s}X_1(s)$, we have $x(t) = x_1(t-1)$

$\therefore x(t) = e^{-(t-1)}u(t-1) + (t-1)e^{-(t-1)}u(t-1)$

(g) $X(s) = \dfrac{2s^2-6s+3}{s^2-3s+2} = 2 - \dfrac{1}{s-1} + \dfrac{1}{s-2}$

$\therefore x(t) = 2\delta(t) - e^{t}u(t) + e^{2t}u(t)$

(h) $X(s) = \dfrac{2(s^2+2s+4)}{(s^2+4)^2} = 2 + \dfrac{4s}{(s^2+4)^2}$

$\therefore x(t) = 2\delta(t) + t\sin 2t\, u(t)$

(i) $X(s) = \dfrac{2}{(2s+1)^3} = \dfrac{1}{8}\,\dfrac{2}{(s+\frac{1}{2})^3}$

$\therefore x(t) = \frac{1}{8} t^2 e^{-\frac{1}{2}t} u(t)$

(j) $X(s) = \dfrac{(s^2+8)e^{-2s}}{s(s^2+16)} = \dfrac{\frac{1}{2}e^{-2s}}{s} + \dfrac{\frac{1}{2}s e^{-2s}}{s^2+16}$

$x(t) = \frac{1}{2}u(t-2) + \frac{1}{2}\cos[2(t-2)]\,u(t-2)$

5.12 (a) $Y(s) = \frac{1}{s-a} \cdot \frac{1}{s-b} = \frac{\frac{1}{a-b}}{s-a} + \frac{\frac{1}{b-a}}{s-b}$

$y(t) = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]u(t)$

(b) $Y(s) = \frac{1}{(s-a)^2}$, $y(t) = t\,e^{at}u(t)$

(c) $Y(s) = \left[\frac{e^{s} - e^{-s}}{s}\right]\frac{1}{s}$

$y(t) = (t+1)u(t+1) - (t-1)u(t-1) = \begin{cases} 0 & t \le -1 \\ t+1 & -1 \le t \le 1 \\ 2 & t \ge 1 \end{cases}$

(d) $Y(s) = \frac{1}{s^2} \cdot \frac{1}{s+a} = \frac{1/a}{s^2} - \frac{1/a^2}{s} + \frac{1/a^2}{s+a}$

$y(t) = \frac{1}{a}\,t\,u(t) - \frac{1}{a^2}u(t) + \frac{1}{a^2}e^{-at}u(t)$

(e) $Y(s) = \frac{1}{s+b} \cdot \frac{1}{s} = \frac{-1/b}{s+b} + \frac{1/b}{s}$

$y(t) = \frac{1}{b}u(t) - \frac{1}{b}e^{-bt}u(t)$

(f) $Y(s) = \frac{a}{s^2+a^2} \cdot \frac{s}{s^2+b^2} = \frac{\frac{a}{a^2-b^2}s}{s^2+b^2} - \frac{\frac{a}{a^2-b^2}s}{s^2+a^2}$

$\therefore y(t) = \frac{a}{a^2-b^2}\left[\cos(bt) + \cos(at)\right]u(t)$

(g) $Y(s) = \frac{1}{s+2} \cdot \left[\frac{1-e^{-2s}}{s}\right] = (1-e^{-2s})\left[\frac{1/2}{s} - \frac{1/2}{s+2}\right]$

$y(t) = \frac{1}{2}u(t) - \frac{1}{2}u(t-2) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-2(t-2)}u(t-2)$

(h) $Y(s) = \left[\frac{1}{s+2} + 1\right]\frac{e^{-s}}{s} = \left[\frac{1/2}{s} - \frac{1/2}{s+2} + \frac{1}{s}\right]e^{-s}$

$y(t) = \frac{3}{2}u(t-1) - \frac{1}{2}e^{-2(t-1)}u(t-1)$

5.13 (a) (i) $Y(s) = \frac{1}{s-a} \cdot \frac{1}{s-a}$

$\therefore y(t) = e^{-at}u(t) * e^{-at}u(t) = t\,e^{-at}u(t)$

(ii) $Y(s) = \dfrac{1}{(s-a)^2} \cdot \dfrac{1}{s-a}$

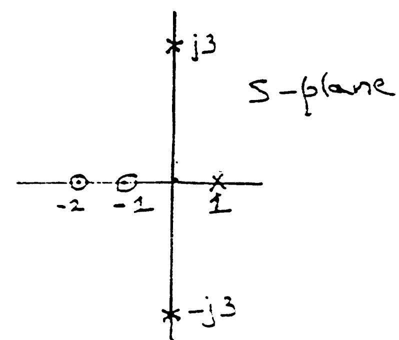
$\therefore y(t) = t e^{-at} u(t) * e^{-at} u(t)$

$= \int_{-\infty}^{\infty} \tau e^{-a\tau} u(\tau)\, e^{-a(t-\tau)} u(t-\tau)\, d\tau$

$= \int_0^t \tau e^{-at}\, d\tau = \dfrac{t^2}{2} e^{-at} u(t)$

(iii) $\mathcal{L}^{-1}\left[\dfrac{1}{(s-a)^n}\right] = \dfrac{t^{n-1}}{(n-1)!} e^{-at} u(t)$

5.14 (a) $H(s) = \dfrac{(s+1)(s+2)}{(s-1)(s^2+9)}$



Pole-zero plot of $H(s)$

(b) $H(s) = \dfrac{3/5}{s-1} + \dfrac{\frac{2}{5}s - 2}{s^2+9}$

$= \dfrac{3/5}{s-1} + \dfrac{2}{5}\dfrac{s}{s^2+9} - \dfrac{2}{3}\dfrac{3}{s^2+9}$

$\therefore h(t) = \dfrac{3}{5} e^{t} u(t) + \dfrac{2}{5} \cos 3t\, u(t) - \dfrac{2}{3} \sin t\, u(t)$

and is real

(c) $H^*(s) = \left[\int_0^{\infty} h(t) e^{-st}\, dt\right]^* = \int_0^{\infty} h^*(t) e^{-s^* t}\, dt$

$= \int_0^{\infty} h(t) e^{-s^* t}\, dt = H(s^*)$

Suppose $s = s_0$ is a zero of $H(s)$ and let

$H(s) = (s - s_0) H_1(s)$

Then $H(s_0) = H^*(s_0) = 0$

But $H^*(s_0) = H(s_0^*)$ so that $H(s_0^*) = 0$

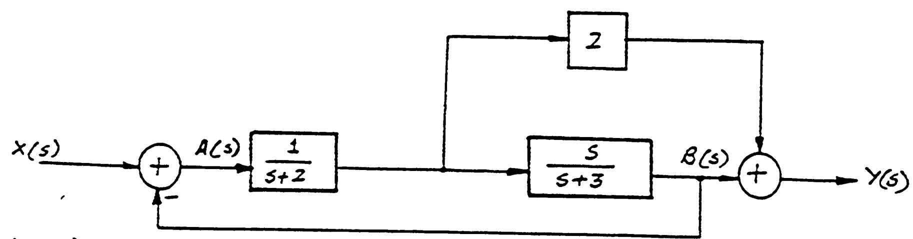
i.e. s_0^* is a zero of $H(s)$

Similarly for a pole.

(d) $H(s)$ has zeros at $-1, -2$ and poles at $1, \pm j3$

That is, poles and zeros occur in complex conjugate pairs.

5.15



$$A(s) = X(s) - B(s)$$

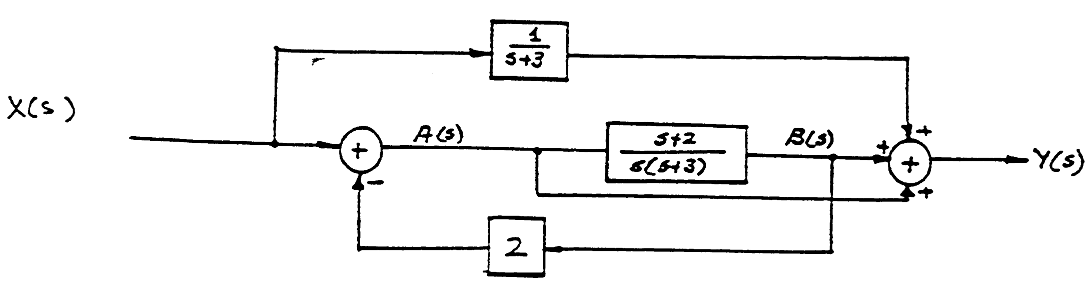
$$B(s) = \frac{s\,A(s)}{(s+2)(s+3)}$$

$$A(s) = X(s) - \frac{s\,A(s)}{(s+2)(s+3)} \quad \text{or} \quad A(s) = \frac{X(s)}{1 + \frac{s}{(s+2)(s+3)}}$$

$$Y(s) = B(s) + \frac{2\,A(s)}{s+2} = \left[\frac{s}{(s+2)(s+3)} + \frac{2}{s+2}\right] A(s)$$

$$= \left[\frac{s}{(s+2)(s+3)} + \frac{2}{s+2}\right] \frac{X(s)}{1 + \frac{s}{(s+2)(s+3)}}$$

$$\text{or} \quad \frac{Y(s)}{X(s)} = \frac{s + 2(s+3)}{(s+2)(s+3)+s} = \frac{3(s+2)}{s^2+6s+6}$$



$$A(s) = X(s) - 2B(s)$$

$$B(s) = \frac{s+2\;A(s)}{s(s+3)}$$

or $$A(s) = X(s) - \frac{2(s+2)A(s)}{s(s+3)} \Rightarrow A(s) = \frac{X(s)}{1 + \frac{2(s+2)}{s(s+3)}}$$

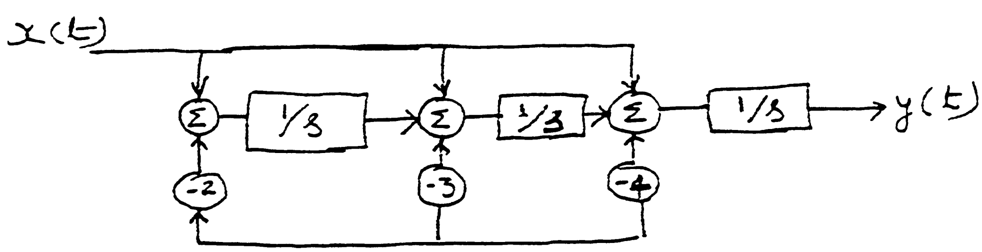
$$B(s) = \frac{(s+2)A(s)}{s(s+3)} = \frac{s+2}{s(s+3)} \cdot \frac{X(s)}{1+\frac{2(s+2)}{s(s+3)}} = \frac{(s+2)X(s)}{s^2+5s+2}$$

$$Y(s) = \frac{1}{s+3}X(s) + B(s) + A(s)$$

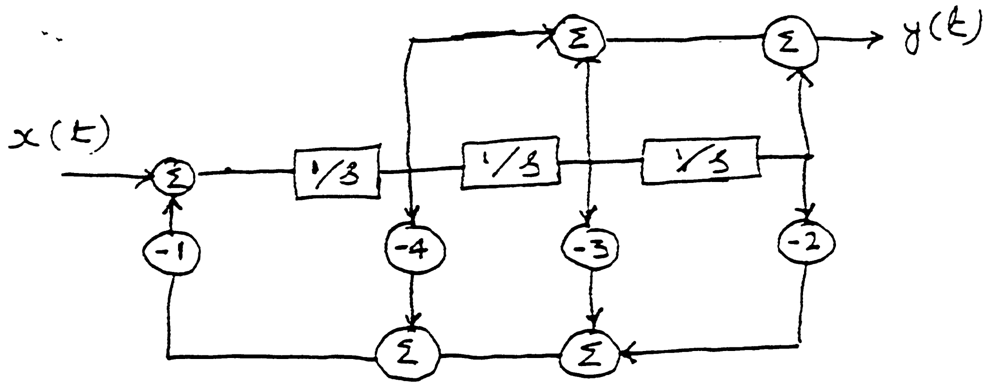
$$= \left[\frac{1}{s+3} + \frac{s+2}{s^2+5s+2} + \frac{s(s+3)}{s^2+5s+2}\right] X(s)$$

or $$\frac{Y(s)}{X(s)} = \frac{s^3+8s^2+19s+8}{(s+3)(s^2+5s+2)}$$

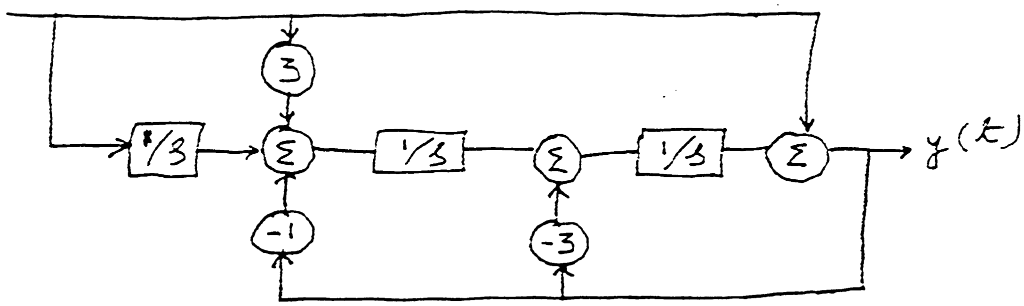
5.16 The first canonical form is



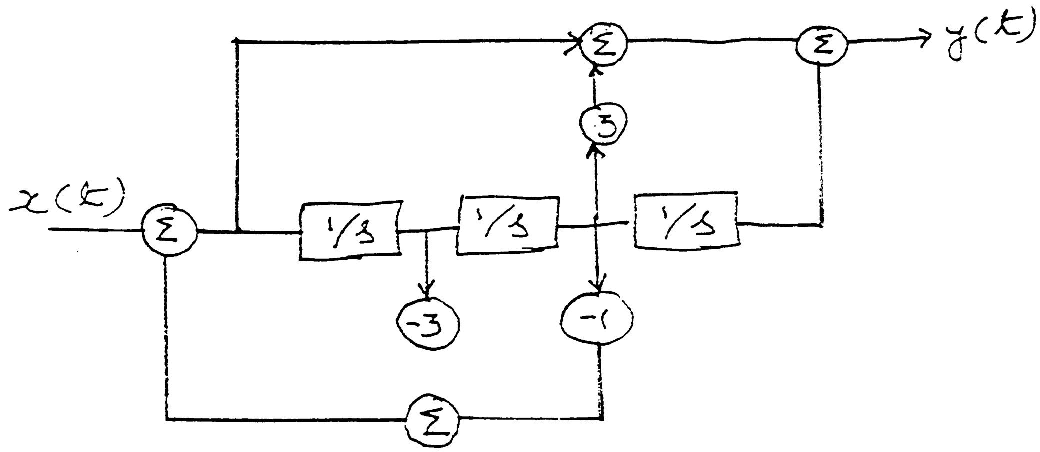
The second canonical form is



5.17 For the first form, we have



The second form is

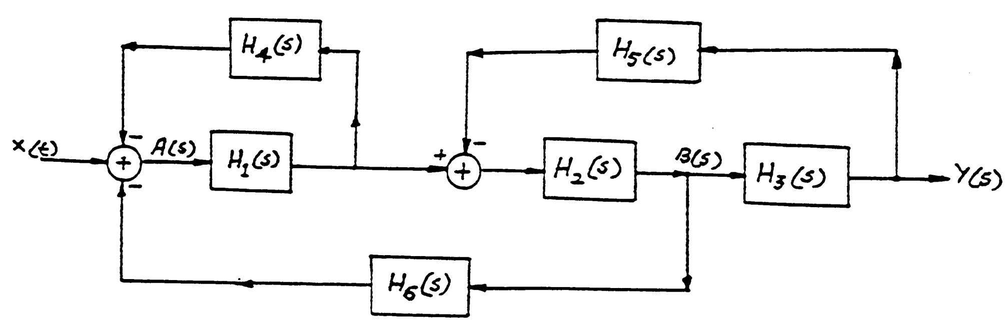


5.18 $2y''(t) + 3y'(t) + 4y(t) = 2x'(t) - x(t)$

$$(2s^2 + 3s + 4)\,Y(s) = (2s-1)\,X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{2s-1}{2s^2+3s+4}$$

5.19



(c) $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$\exp[At] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-1} & \frac{-2}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} \exp[t] & -2t\exp[t] \\ 0 & \exp[t] \end{bmatrix}$$

(d) $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

$$\exp[At] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{s-1}{(s-2)^2+1} & \frac{2}{(s-2)^2+1} \\ \frac{-1}{(s-2)^2+1} & \frac{s-3}{(s-2)^2+1} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[2t][\cos t + \sin t] & 2e^{2t}\sin t \\ -\exp[2t]\sin t & \exp[2t][\cos t - \sin t] \end{bmatrix}$$

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

$$\exp[At] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{-(s+1)}{(s-1)^2 s} & \frac{1}{s-1} & \frac{1}{s(s-1)} \\ \frac{-1}{(s-1)s} & 0 & -\frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[t] & 0 & 0 \\ \exp[t] - 2t\exp[t] - u(t) & \exp[t] & \exp[t] - u(t) \\ -\exp[t] + u(t) & 0 & u(t) \end{bmatrix}$$

$$A(s) = X(s) - H_1(s)\,H_4(s)\,A(s) - H_6(s)\,B(s)$$

$$\text{or} \quad \left[1 + H_1(s)\,H_4(s)\right] A(s) = X(s) - H_6(s)\,B(s) \qquad \text{---}(1)$$

$$B(s) = H_1(s)\,H_2(s)\,A(s) - H_3(s)\,H_5(s)\,H_2(s)\,B(s)$$

$$\text{or} \quad (1 + H_3 H_5)\,B(s) = H_1(s)\,H_2(s)\,A(s)$$

$$\text{or} \quad A(s) = \frac{1 + H_3(s)\,H_5(s)}{H_1(s)\,H_2(s)}$$

$$\text{from (1)} \quad \frac{(1 + H_1(s)\,H_4(s))\,(1 + H_3(s)\,H_5(s)\,H_2(s))\,B(s)}{H_1(s)\,H_2(s)}$$

$$= X(s) - H_6(s)\,B(s)$$

$$\text{or} \quad B(s)\left[\frac{(1 + H_1(s)\,H_4(s))\,(1 + H_3(s)\,H_5(s)\,H_2(s)) + H_6(s)\,H_1(s)\,H_2(s)}{H_1(s)\,H_2(s)}\right]$$

$$= X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_3(s)\,B(s)}{X(s)}$$

$$= \frac{H_1(s)\,H_2(s)\,H_3(s)}{(1 + H_1(s)\,H_4(s))(1 + H_3(s)\,H_5(s)\,H_2(s)) + H_6(s)\,H_1(s)\,H_2(s)}$$

5.20 (a) $y'(t) + 2y(t) = u(t) \qquad y(0) = 1$

$$sY(s) - y(0) + 2Y(s) = u(s)$$

$$\Longrightarrow \quad (s+2)\,Y(s) = 1 + \frac{1}{s} \quad \text{or} \quad Y(s) = \frac{1}{s+2} + \frac{1}{s(s+2)}$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2}u(t) + \frac{1}{2}\exp[-2t]$$

(b) $y'(t) + 2y(t) = \cos t\; u(t) \qquad y'(0) = 1$

$$sY(s) - y'(0) + 2Y(s) = \frac{s}{s^2+1}$$

$$\Longrightarrow \quad Y(s) = \frac{1}{s+2} + \frac{s}{(s^2+1)(s+2)} = \frac{\frac{3}{5}}{s+2} + \frac{\frac{2}{5}\left(s+\frac{1}{2}\right)}{s^2+1}$$

$\therefore\ y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{5}\exp[-2t] + \frac{2}{5}\cos t + \frac{1}{5}\sin t$

(c) $y'(t) + 2y(t) = \exp[-3t]u(t) \qquad y(0) = 1$

$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$

$\Longrightarrow\ Y(s) = \frac{1}{s+3} + \frac{1}{(s+2)(s+3)}$

$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{2}\exp[-2t] - \frac{1}{2}\exp[-3t]$

(d) $y''(t) + 4y'(t) + 3y(t) = u(t) \qquad y(0)=2 \qquad y'(0)=1$

$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{1}{s}$

$\Longrightarrow (s^2+4s+3)Y(s) = 2s+9 + \frac{1}{s}$

$Y(s) = \frac{2s+9}{s^2+4s+3} + \frac{1}{s(s^2+4+3)} = \frac{3}{s+1} + \frac{\frac{1}{3}}{s} + \frac{-\frac{4}{3}}{s+3}$

$y(t) = \mathcal{L}^{-1}[Y(s)] = 3\exp[-t] + \frac{1}{3}u[t] - \frac{4}{3}\exp[-3t]$

(e) $y''(t) + 4y'(t) + 3y(t) = \exp[-3t]u(t)$

$s^2Y(s) - sy'(0) - y(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{1}{s+3}$

$(s^2+4s+3)Y(s) = 5y(0) + sy'(0) + \frac{1}{s+3}$

$Y(s) = \frac{5y(0)+sy'(0)}{s^2+4s+3} + \frac{1}{(s+3)(s^2+4s+3)}$

$= \frac{\frac{5}{2}y(0) - \frac{1}{2}y'(0)}{s+1} + \frac{\frac{5}{-2}y(0) + \frac{3}{2}y'(0)}{s+3} + \frac{\frac{1}{4}}{s+1} - \frac{1}{4}\frac{s+5}{(s+3)^2}$

$= \frac{\frac{5}{2}y(0) - \frac{1}{2}y'(0) + \frac{1}{4}}{s+1} + \frac{-\frac{5}{2}y(0) + \frac{3}{2}y'(0) - \frac{1}{4}}{s+3} - \frac{\frac{1}{2}}{(s+3)^2}$

$\Longrightarrow\ y(t) = \mathcal{L}^{-1}[Y(s)] = \left(\frac{5}{2}y(0) - \frac{1}{2}y'(0) + \frac{1}{4}\right)\exp[-t]$

$+ \left(-\frac{5}{2}y(0) + \frac{3}{2}y'(0) - \frac{1}{4}\right)\exp[-3t] - \frac{1}{2}t\exp[-3t]$

(f) $y'''(t) + 3y''(t) + 2y'(t) - 6y(t) = \exp[-2].u(t)$,

$y(0) = y'(0) = y''(0) = 0$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 3[s^2Y(s) - sy(0) - y'(0)]$$

$$+ 2[sY(s) - y(0)] + sY(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s^3+3s^2+2s-6)(s+2)} = \frac{1}{(s-1)(s+2)(s^2+4s+6)}$$

$$= \frac{\frac{1}{33}}{s-1} + \frac{-\frac{1}{6}}{s+2} + \frac{\frac{3}{22}s + \frac{2}{11}}{s^2+4s+6}$$

$$y(t) = \frac{1}{33}\exp[t] - \frac{1}{6}\exp[-2t] + \frac{3}{22}\exp[-2t]\cos\sqrt{2}t - \frac{1}{11\sqrt{2}}\exp[-2t]\sin\sqrt{2}t$$

5.21 (a) $x(t) = \delta(t)$

$$y'(t) + 5y(t) = x(t) + 2x'(t)$$

$$Y(s) = \frac{2s+1}{s+5} = 2 + \frac{-9}{s+5}$$

$$h(t) = \mathcal{L}^{-1}[Y(s)] = 2\delta(t) - 9\exp[-5t]$$

(b) $y''(t) + 4y'(t) + 3y(t) = 2x(t) - 3x'(t)$ $\qquad x(t) = \delta(t)$

$$H(s) = \frac{2-3s}{s^2+4s+3} = \frac{\frac{5}{2}}{s+1} + \frac{-\frac{11}{2}}{s+3}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{5}{2}\exp[-t] - \frac{11}{2}\exp[-3t]$$

(c) $y'''(t) + y'(t) - 2y(t) = x''(t) + x'(t) + 2x(t)$ $\qquad x(t) = \delta(t)$

$$H(s) = \frac{s^2+s+2}{s^3+s-2} = \frac{1}{s-1}$$

$$h(t) = \mathcal{L}[H(s)]$$

$$= \exp[t]\,u(t)$$

5.22 (a) $X(s) = \frac{2}{s+2}$, $Y(s) = -\frac{1}{s^2} + \frac{1}{s+2} + \frac{1}{s+1} + \frac{1}{s}$

$$= \frac{3s^3 + 5s^2 - s - 2}{s^2(s+1)(s+2)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s^3 + 5s^2 - s - 2}{2s^2(s+1)} = \frac{3}{2} + \frac{s^2 - \frac{5}{4}s - 1}{(s^2)(s+1)}$$

$$= \frac{3}{2} + \frac{\frac{5}{4}}{s+1} + \frac{-\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s^2}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{3}{2}\delta(t) + \frac{5}{4}\exp[-t]u(t) - \frac{1}{4}u(t) - \frac{1}{4}tu(t)$$

(b) $X(s) = \frac{2}{s}$ $\qquad Y(s) = \frac{1}{s^2} - \frac{1}{s+2}$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s^2} - \frac{1}{s+2}}{\frac{2}{s}} = \frac{-s^2 + s + 2}{2s(s+2)} = -\frac{1}{2} + \frac{\frac{1}{2}}{s} + \frac{1}{s+2}$$

$$\Longrightarrow h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{2}\delta(t) + \frac{1}{2}u(t) + \exp[-2t]u(t)$$

(c) $X(s) = \frac{1}{s+2}$, $Y(s) = \frac{1}{s+1} + \frac{-3}{s+2}$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1} + \frac{-3}{s+2}}{\frac{1}{s+2}} = \frac{s+2}{s+1} - 3 = -2 + \frac{1}{s+1}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -2\delta(t) + \exp[-t]\,u(t)$$

(d) $X(s) = \frac{1}{s^2}$ $\qquad Y(s) = \frac{2}{s^3} - \frac{2}{s+3}$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{2}{s^3} - \frac{2}{s+3}}{\frac{1}{s^2}} = \frac{2}{s} - \frac{2s^2}{s+3}$$

$$= \frac{2}{s} - 2s + 6 - \frac{18}{s+3}$$

$$\therefore h(t) = 2u(t) - 2\delta'(t) + 6\delta(t) - 18\exp[-3t]\,u(t)$$

(e) $X(s) = \frac{2}{s}$ $\qquad Y(s) = -\frac{1}{\sqrt{2}}\left(\frac{s+2}{(s+2)^2+16} + \frac{4}{(s+2)^2+16}\right)$

$$= -\frac{1}{\sqrt{2}}\,\frac{s+6}{(s+2)^2+16}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{-\frac{1}{\sqrt{2}}\left(\frac{s+6}{(s+2)^2+16}\right)}{\frac{2}{s}} = -\frac{1}{2\sqrt{2}}\,\frac{s^2+6s}{(s+2)^2+16}$$

$$= -\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\,\frac{s-10}{(s+2)^2+16}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{2\sqrt{2}}\,\delta(t) - \frac{1}{\sqrt{2}}\exp[-2t]\cos 4t$$

$$+ \frac{3}{\sqrt{2}}\exp[-2t]\sin 4t$$

(f) $X(s) = \frac{3}{s^2}$ $\quad , Y(s) = -\frac{3}{\sqrt{2}}\,\frac{s+4}{(s+4)^2+16} + \frac{1}{\sqrt{2}}\,\frac{4}{(s+4)^2+16}$

$$= \frac{1}{\sqrt{2}}\,\frac{-3s-8}{(s+4)^2+16}$$

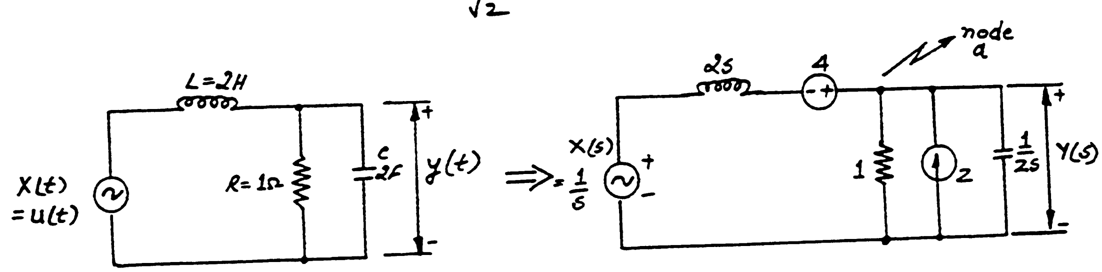
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{\sqrt{2}}\,\frac{-3s-8}{(s+4)^2+16}}{3/s^2} = \frac{(-3s-8)\,s^2}{3\sqrt{2}\,[(s+4)^2+16]}$$

$$= \frac{-3s+16}{3\sqrt{2}} + \frac{-32s-512}{(s+4)^2+16}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{\sqrt{2}}\,\delta'(t) + \frac{16}{3\sqrt{2}}\,\delta(t) - \frac{32}{3\sqrt{2}}\exp[-4t]\cos 4t$$

$$-\frac{32}{\sqrt{2}}\exp[-4t]\sin 4t$$

5.23



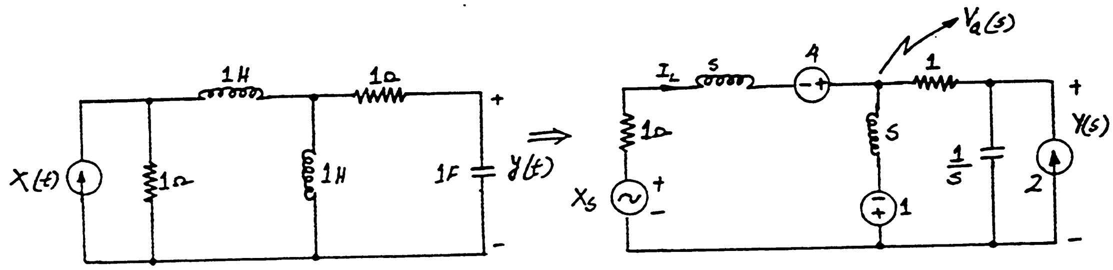
at node a, $$\frac{\frac{1}{s} + 4 - Y(s)}{2s} + 2 - Y(s) - Y(s)\,2s = 0$$

$$\Longrightarrow \quad \frac{1}{s} + 4 - Y(s) + 4s = (1+2s)^2 \cancel{8}\, Y(s)$$

$$\Longrightarrow \quad Y(s) = \frac{s^2 + s + 1/4}{s\,(s^2 + \frac{1}{2}s + 1/4)} = \frac{1}{s} + \frac{\frac{1}{2}}{s^2 + \frac{1}{2}s + 1/4}$$

$$y(t) = \mathcal{L}\left[Y(s)\right] = u(t) + \frac{2}{\sqrt{5}} \exp[-\tfrac{1}{4}t] \sin \frac{\sqrt{5}}{4} t$$

5.24



use KVL

$$\frac{X + 4 - V_a}{1+s} = V_a - Y(s) + \frac{V_a + 1}{s}$$

or $$X + 4 = \left[V_a - Y(s)\right](1+s) + \frac{V_a + 1}{s} \qquad \text{———} \quad (1)$$

$$V_a - Y(s) = -2 + sY(s)$$

$$V_a = -2 + (1+s)Y(s) \qquad \text{———} \quad (2)$$

(2) substitute into (1)

$$\Longrightarrow \quad Y(s) = \frac{s\,X(s) + 2s^2 + 9s + 1}{(s+1)^3}$$

$$X(t) = u(t) \Longrightarrow X(s) = \frac{1}{s}$$

$$Y(s) = \frac{2s^2 + 9s + 2}{(s+1)^3} = \frac{2(s+1)^2 + 5(s+1) - 5}{(s+1)^3}$$

$$= \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{5}{(s+1)^3}$$

$$\therefore \quad y(t) = \left(2\exp[-t] + 5t\exp[-t] - \frac{5}{2}t^2\exp[-t]\right)u(t)$$

5.25 $x(t) = \cos t\, u(t) \qquad X(s) = \dfrac{s}{s^2+1}$

$$\Rightarrow \frac{\dfrac{s}{s^2+1} + 4 - Y(s)}{2s} + 2 - Y(s) - 2s\,Y(s) = 0$$

$$\text{or} \quad \frac{s}{s^2+1} + 4 - Y(s) + 4s = 2s(1+2s)\,Y(s)$$

$$\Rightarrow Y(s) = \frac{\dfrac{s}{s^2+1} + 4s + 4}{4s^2 + 2s + 1} = \frac{s}{4(s^2+1)\left[(s+\frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2\right]}$$

$$+ \frac{s+1}{(s+\frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2}$$

$$= \frac{-\frac{3}{13}s + \frac{2}{13}}{4(s^2+1)} + \frac{\frac{12}{13}s - \frac{2}{13} + s + 1}{4\left[(s+\cancel{\frac{1}{4}})^2 + (\frac{\sqrt{3}}{4})^2\right]}$$

$$\Rightarrow y(t) = -\frac{3}{\sqrt{2}}\cos t + \frac{2}{52}\sin t + \frac{25}{52}\exp[-\tfrac{1}{4}t]\cos\frac{\sqrt{3}}{4}t$$

$$+ \frac{19}{52\sqrt{3}}\sin\frac{\sqrt{3}}{4}t$$

5.26 $x(t) = \sin 2t\; u(t) \qquad X(s) = \dfrac{2}{s^2+4}$

$$\text{from } 5.24 \quad Y(s) = \frac{\dfrac{2s}{s^2+4} + 2s^2 + 9s + 1}{(s+1)^3}$$

$$= \frac{2s}{(s^2+4)(s+1)^3} + \frac{2s^2+9s+1}{(s+1)^3}$$

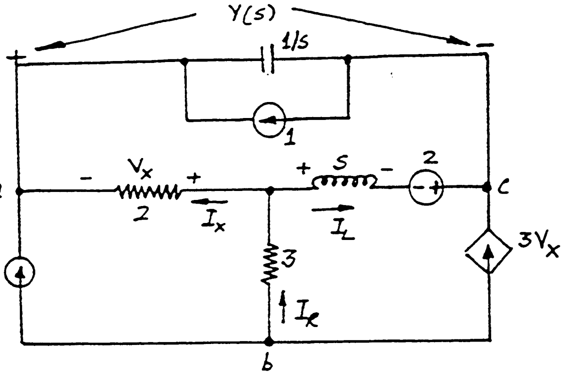
$$= \frac{\frac{154}{37}s - \frac{56}{37}}{s^2+4} + \frac{\frac{154}{37}s^2 + \frac{1}{37}s + \frac{14}{37} + 2s^2 + 9s + 1}{-(s+1)^3}$$

$$= \frac{-\frac{154}{37}s - \frac{56}{37}}{s^2+4} + \frac{\frac{1}{37}\left[154(s+1)^2 + 124(s+1) - 227\right]}{(s+1)^3}$$

$$\Rightarrow y(t) = \Big(-\frac{154}{37}\cos 2t - \frac{28}{37}\sin 2t + \frac{154}{37}\exp[-t]$$

$$+ \frac{124}{37}t\exp[-t] - \frac{227}{74}t^2\exp[-t]\Big)u(t)$$

5.27



at node a: $\frac{1}{s} + \frac{V_x}{2} + 1 - sY = 0$

at node b: $\frac{1}{s} + 3V_x + I_R = 0$

$$V_x + 2 + y(s) - I_L s = 0$$

$$I_L + \frac{V_x}{2} = I_R \implies \frac{1}{s} + \frac{7}{2} V_x + I_L = 0$$

$$\implies \begin{cases} \frac{V_x}{2} - sY = -1 - \frac{1}{s} \\ V_x - sI_L + Y(s) = -2 \\ \frac{7}{2} V_x + I_L = -\frac{1}{s} \end{cases}$$

$$\begin{bmatrix} \frac{7}{2} & 1 & 0 \\ 1 & -s & 1 \\ \frac{1}{2} & 0 & -s \end{bmatrix} \begin{bmatrix} V_x \\ I_L \\ Y(s) \end{bmatrix} = \begin{bmatrix} -\frac{1}{s} \\ -2 \\ -1-\frac{1}{s} \end{bmatrix}$$

$$\implies Y(s) = \frac{\frac{7}{2}s + 3 + \frac{1}{s}}{\frac{7}{2}s^2 + s + \frac{1}{2}} = \frac{-7s+2}{7s^2+2s+1} + \frac{2}{s}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 2u(t) - \exp[-\tfrac{1}{7}t] \cos \tfrac{\sqrt{48}}{7} t\, u(t) + \frac{3}{\sqrt{48}} \exp[-\tfrac{1}{7}t] \sin \tfrac{\sqrt{48}}{7} t\, u(t)$$

5.28 (a) $Y(s) = \dfrac{H_1(s)\,H_2(s)}{1 + H_1(s) + H_2(s)} = \dfrac{29\,\frac{1}{s(s+a)}}{1 + 29 \cdot \frac{1}{s\cdot(s+a)}} = \dfrac{29}{s^2 + as + 29}$

(b) (i) $a = 5$, $X(s) = \frac{1}{s}$

$$Y(s) = \frac{29}{s(s^2+5s+29)} = \frac{1}{s} + \frac{-s-5}{s^2+5s+29} = \frac{1}{s} + \frac{-(s+\frac{5}{2})-\frac{5}{2}}{(s+\frac{5}{2})^2+\frac{91}{4}}$$

$$\therefore\ y(t) = u(t) - \exp[-\tfrac{5}{2}t]\cos\frac{\sqrt{91}}{2}t - \frac{5}{\sqrt{91}}\exp[-\tfrac{5}{2}t]\sin\frac{\sqrt{91}}{2}t$$

(ii) $a = 3$

$$Y(s) = \frac{29}{s(s^2+3s+29)} = \frac{1}{s} + \frac{-s-3}{s^2+3s+29}$$

$$= \frac{1}{s} + \frac{-(s+\frac{3}{2})-\frac{3}{2}}{(s+\frac{3}{2})^2+\left(\frac{\sqrt{107}}{2}\right)^2}$$

$$y(t) = u(t) - \exp[-\tfrac{3}{2}t]\cos\frac{\sqrt{107}}{2}t - \frac{3}{\sqrt{107}}\exp[-\tfrac{3}{2}t]\sin\frac{\sqrt{107}}{2}t$$

(iii) $a = 1$

$$Y(s) = \frac{29}{s(s^2+s+29)} = \frac{1}{s} + \frac{-s-1}{s^2+s+29} = \frac{1}{s} + \frac{-(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{115}{4}}$$

$$y(t) = u(t) - \exp[-\tfrac{t}{2}]\cos\frac{\sqrt{115}}{2}t - \frac{1}{\sqrt{115}}\exp[-\tfrac{1}{2}t]\sin\frac{\sqrt{115}}{2}t$$

5.29 $$\frac{Y(s)}{X(s)} = \frac{H_c(s)H(s)}{1+H_c(s)H(s)}$$

$X(s) = \frac{A}{s}$ $\qquad H_c(s) = \frac{s+1}{s}$, $\quad H(s) = \frac{1}{s+2}$

(a) $$Y(s) = \frac{H_c(s)H(s)}{1+H_c(s)H(s)} = \frac{\frac{s+1}{s}\cdot\frac{1}{s+2}\cdot\frac{A}{s}}{1+\frac{s+1}{s}\cdot\frac{1}{s+2}}$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\frac{\frac{s+1}{s+2}\left(\frac{A}{s}\right)}{1+\frac{s+1}{(s+2)s}} = \lim_{s\to 0}\frac{(s+1)A}{(s+2)s+s+1} = A$$

(b) $$Y(s) = \frac{\frac{(s+1)A}{s}}{s(s+2)+s+1} = \frac{(s+1)A}{s(s^2+3s+1)} = \frac{A}{s} + \frac{(-s-2)A}{s^2+3s+1}$$

$$= \frac{A}{s} + \frac{\frac{-1-\sqrt{5}}{2\sqrt{5}}}{s-\frac{-3+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{s-\frac{-3-\sqrt{5}}{2}}$$

$$y(t) = A\,u(t) - \frac{(1+\sqrt{5})A}{2\sqrt{5}} \exp\left[\left(-\frac{3+\sqrt{5}}{2}\right)t\right] + \frac{(1-\sqrt{5})A}{2\sqrt{5}} \exp\left[\left(\frac{-3-\sqrt{5}}{2}\right)t\right]$$

$$\exp[t] = x(t) - y(t) = \frac{(1+\sqrt{5})A}{2\sqrt{5}} \exp\left[-\frac{3+\sqrt{5}}{2} t\right]$$

$$- \frac{(1-\sqrt{5})A}{2\sqrt{5}} \exp\left[-\frac{3-\sqrt{5}}{2} t\right]$$

(c) No, pole of $H(s)$ is in right-hand side

(d) No one of the zeros of $H_c(s)$ must be at the origin

5.3a (a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ $\quad \exp[At] = \mathcal{L}^{-1}[sI-A]^{-1}$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s-2}{(s-1)^2} & \frac{-1}{(s-1)^2} \\ \frac{1}{(s-1)^2} & \frac{s}{(s-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[t] - t\exp[t] & -t\exp[t] \\ t\exp[t] & \exp[t] + t\exp[t] \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$$\exp[At] = \mathcal{L}^{-1}[sI-A]^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{s}{s^2-s+2} & \frac{-1}{s^2-s+2} \\ \frac{2}{s^2-s+2} & \frac{s-1}{s^2-s+2} \end{bmatrix}$$

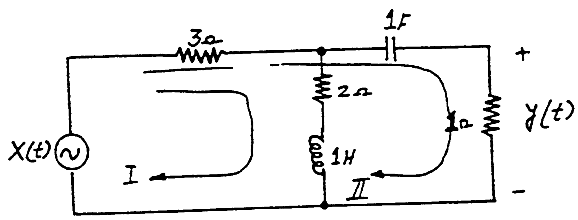
$$= \begin{bmatrix} \exp\left[\frac{t}{2}\right]\cos\frac{\sqrt{7}}{2}t + \frac{1}{\sqrt{7}}\exp\left[\frac{t}{2}\right]\sin\frac{\sqrt{7}}{2}t, & \frac{-2}{\sqrt{7}}\exp\left[\frac{t}{2}\right]\sin\frac{\sqrt{7}}{2}t \\ \frac{4}{\sqrt{7}}\exp\left[\frac{t}{2}\right]\sin\frac{\sqrt{7}}{2}t, & \exp\left[\frac{t}{2}\right]\cos\frac{\sqrt{7}}{2}t + \frac{3}{\sqrt{7}}\exp\left[\frac{t}{2}\right]\sin\frac{\sqrt{7}}{2}t \end{bmatrix}$$

(f) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\exp[At] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-2)^2} & \frac{1}{(s-2)^2} \\ 0 & \frac{s-1}{(s-2)^2} & \frac{1}{(s-2)^2} \\ 0 & \frac{1}{(s-2)^2} & \frac{s-3}{(s-2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[2t] & t\exp[2t] & t\exp[2t] \\ 0 & \exp[2t]+t\exp[2t] & t\exp[2t] \\ 0 & t\exp[2t] & \exp[2t]-t\exp[2t] \end{bmatrix}$$

5.31



(a) loop I : $x(t) = (i_L + i_c)3 + 2i_L + \frac{di_L}{dt}$ .... (1)

loop II : $x(t) = (i_L + i_c)3 + V_c + i_c$ --- (2)

from (1) $x(t) = 5i_L + \frac{3dV_c}{dt} + \frac{di_L}{dt}$ --- (3)

from (2) $x(t) = 3i_L + V_c + 4\frac{dV_c}{dt}$ --- (4)

from (3) & (4) $\frac{di_L}{dt} = \frac{-11}{4}i_L + \frac{3}{4}V_c + \frac{x}{4}$

$\frac{dV_c}{dt} = -\frac{3}{4}i_L - \frac{1}{4}V_c + \frac{x}{4}$

$$\Longrightarrow \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dV_c}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-11}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{-1}{4} \end{bmatrix}}_{A} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \underbrace{\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}}_{b}$$

(b) $\exp[At] = \mathcal{L}^{-1}[(sI-A)^{-1}]$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+1/4}{(s+1/2)(s+5/2)} & \dfrac{3/4}{(s+1/2)(s+5/2)} \\ \dfrac{-3/4}{(s+1/2)(s+5/2)} & \dfrac{s+11/4}{(s+1/2)(s+5/2)} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8}\exp[-\frac{1}{2}t] + \frac{9}{8}\exp[-\frac{5}{2}t] & \frac{3}{8}\exp[-\frac{1}{2}t] - \frac{3}{8}\exp[-\frac{5}{2}t] \\ -\frac{3}{8}\exp[-\frac{1}{2}t] + \frac{3}{8}\exp[-\frac{5}{2}t] & \frac{9}{8}\exp[-\frac{1}{2}t] - \frac{1}{8}\exp[-\frac{5}{2}t] \end{bmatrix}$$

(c) $X(t) = u(t) \qquad V(0^-) = 0 \qquad X(s) = \frac{1}{s}$

$$\begin{bmatrix} I_L(s) \\ V_c(s) \end{bmatrix} = [sI - A]^{-1} b\, X(s)$$

$$= \begin{bmatrix} \dfrac{s+\frac{1}{4}}{(s+1/2)(s+5/2)} & \dfrac{3/4}{(s+\frac{1}{2})(s+5/2)} \\ \dfrac{-3/4}{(s+1/2)(s+5/2)} & \dfrac{s+11/4}{(s+1/2)(s+5/2)} \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} \frac{1}{s}$$

$$V_c(s) = \frac{1}{4}\,\frac{(s+2)\frac{1}{s}}{(s+1/2)(s+5/2)}$$

$$y(t) = i_c = \frac{dV_c}{dt} \Rightarrow Y(s) = s\,V_c(s) - V_c(0) = s\,V_c(s)$$

$$y(t) = \mathcal{L}^{-1}[s\,V_c(s)] = \mathcal{L}^{-1}\left[\frac{1}{4}\,\frac{s+2}{(s+\frac{1}{2})(s+\frac{5}{2})}\right]$$

$$= \left(\frac{3}{16}\exp[-\frac{t}{2}] + \frac{1}{16}\exp[-\frac{5}{2}t]\right)u(t)$$

5.32

(a) $\underline{V}(s) = [sI - A]^{-1}\,\underline{v}(0^-)$

$$= \begin{bmatrix} s+1 & 0 \\ 3 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 0 \\ -3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/(s+1) \\ -3/(s+1) + 3/(s+2) \end{bmatrix}$$

$$\therefore \ \underline{v}(t) = \begin{bmatrix} e^{-t} u(t) \\ -3e^{-t} u(t) + 3e^{-2t} u(t) \end{bmatrix}$$

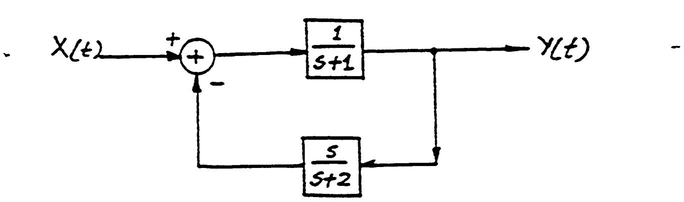
(b) $$\underline{V}(s) = \begin{bmatrix} s-2 & 7 \\ -1 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{s^2+s+1} \begin{bmatrix} s+3 & -7 \\ 1 & s-2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{7}{s^2+s+1} \\ \dfrac{s+2}{s^2+s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 14 \times \dfrac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ -\dfrac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + 5 \dfrac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \end{bmatrix}$$

$$= \begin{bmatrix} 14\, e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t\, u(t) \\ -e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t\, u(t) + 5 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t\, u(t) \end{bmatrix}$$

5.33 (a)

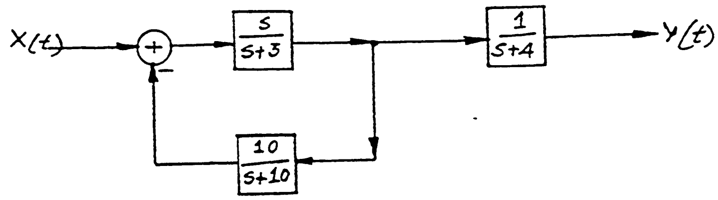


$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1} \cdot \frac{s}{s+2}} = \frac{s+2}{s^2+4s+2}$$

The two poles of $H(s)$, $s_1 = -3.414$, $s_2 = -0.5858$

$\Longrightarrow$ stable

(b)



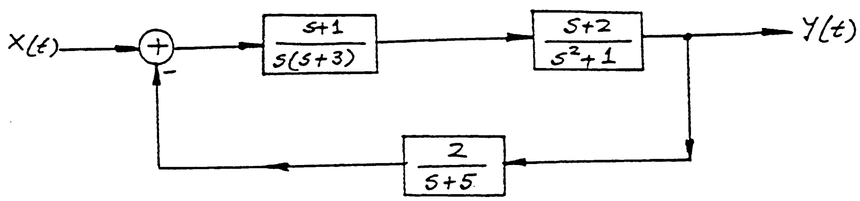
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s}{s+3}\ \frac{1}{s+4}}{1 + \frac{s}{s+3}\ \frac{10}{s+10}} = \frac{s(s+10)}{s^3 + 27s^2 + 122s + 120}$$

the poles of $H(s)$

$s_1 = -1.388$, $s_2 = -21.6$, $s_3 = -4$

are all in the left-side plane $\Longrightarrow$ stable

(c)



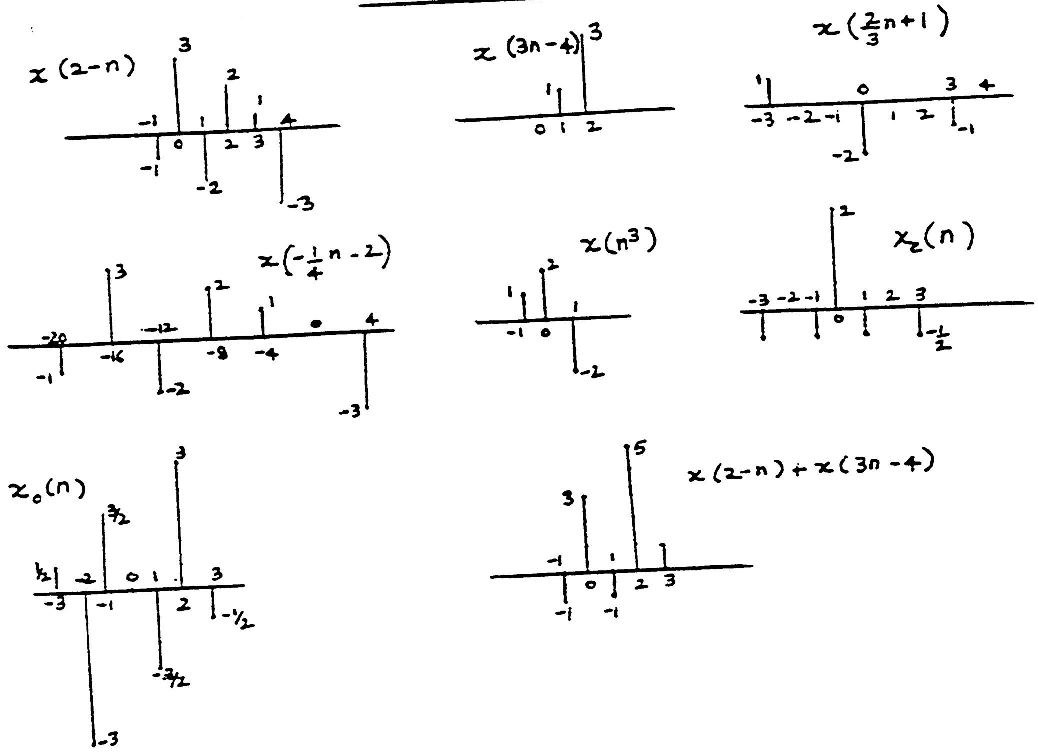
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s+1}{s(s+3)}\left(\frac{s+2}{s^2+1}\right)}{1 + \frac{(s+1)}{s(s+3)} \cdot \frac{(s+2)}{(s^2+1)}\ \frac{2}{(s+5)}}$$

$$= \frac{(s+1)(s+5)(s+2)}{s^5 + 8s^4 + 16s^3 + 10s^2 + 21s + 4}$$

Since these are two poles in the right-hand plane the system is unstable.

CHAPTER 6

6.1



6.2 (a) $x(2-n) = \{-1, -\frac{3}{2}, 1, 0, \frac{1}{3}, -1\}$

(b) $x(3n-4) = \{0, -1, 1\}$

(c)

$$x(\frac{2}{3}n+1) = \begin{cases} -1 & n=-2,\ 11 \\ 1/3 & n=0 \\ 1 & n=5 \\ -3/2 & n=8 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$x(-\frac{n+8}{4}) = \begin{cases} -1 & n=-24, -4 \\ -3/2 & n=-20 \\ 1 & n=-16 \\ 1/3 & n=-8 \\ 0 & \text{otherwise} \end{cases}$$

(e)

$$x(n^3) = \begin{cases} -1 & n=-1 \\ \frac{1}{3} & n=0 \\ 1 & n=8 \\ -\frac{3}{2} & n=27 \\ -1 & n=64 \\ 0 & \text{otherwise} \end{cases}.$$

(f) $x_e(n) = \dfrac{x(n)+x(-n)}{2} = \{ \dfrac{-1}{2}, -\dfrac{3}{4}, \dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{1}{2}, -\dfrac{3}{4}, -\dfrac{1}{2} \}$

(g) $x_o(n) = \dfrac{x(n)-x(-n)}{2} = \{ \dfrac{1}{2}, \dfrac{3}{4}, -\dfrac{1}{2}, -\dfrac{1}{2}, 0, \dfrac{1}{2}, \dfrac{1}{2}, -\dfrac{3}{4}, -\dfrac{1}{2} \}$

(h) $x(2-n)+x(3n-4) = \{ -1, -\dfrac{3}{2}, 1, -1, \dfrac{4}{3}, -1 \}$

6.3 (a) $\dfrac{\Omega_0}{2\pi} = \dfrac{\pi/4}{2\pi} = \dfrac{1}{8}$; $N = 8$

(b) $\dfrac{\Omega_{01}}{2\pi} = \dfrac{3\pi/4}{2\pi} = \dfrac{3}{8}$; $N_1 = 8$; $\dfrac{\Omega_{02}}{2\pi} = \dfrac{\pi/3}{2\pi} = \dfrac{1}{6}$; $N_2 = 6$

$x(n)$ is periodic with $N = 24$.

(b) $x(n) = \dfrac{1}{2}\cos\left(\dfrac{13\pi}{12}n\right) - \dfrac{1}{2}\cos\left(\dfrac{5\pi}{12}n\right)$

so that

$\dfrac{\Omega_{01}}{2\pi} = \dfrac{13\pi/12}{2\pi} = \dfrac{13}{24}$; $N_1 = 24$; $\dfrac{\Omega_{02}}{2\pi} = \dfrac{5\pi/12}{2\pi} = \dfrac{5}{24}$; $N_2 = 24$

$x(n)$ is periodic with $N = 24$.

(d) Not periodic

(e) $\dfrac{\Omega_0}{2\pi} = \dfrac{5\pi/6}{2\pi} = \dfrac{5}{12}$; $N = 12$

(f) $N_1 = 2$, $N_2 = 3$ so that $N = 6$

(g) $\frac{\Omega_{01}}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8;} N_1 = 8; \ \frac{\Omega_{02}}{2\pi} = \frac{\pi/3}{2\pi} = \frac{1}{6}; N_2 = 6$

so that $N = 24$.

6.4 $x(n) = x_o(nT) = 5\cos 120(\pi T n - \frac{\pi}{3})$

$\therefore \Omega_o = 120\pi T \ , \ N = \frac{2\pi m}{120\pi T} = \frac{Tm}{60}$

Periodic for any integer T

6.5 $x(n) = 3\sin 100 n\pi T + 4\cos 120 n\pi T$

$N_1 = \frac{2\pi m}{100\pi T} = \frac{m}{50T} \ ; \ N_2 = \frac{2\pi l}{120\pi T} = \frac{l}{60T}$ for any integer T.

$N = pN_1 = qN_2$ so that $p = \frac{5ql}{6m}$

Periodic for any integer T

6.6 Equality (i) for $\alpha \neq 1$ and equalities (ii) and (iii) can be established by dividing the denominator into the numerator by long division

For $\alpha = 1$, left side of equality (i) becomes

$$\sum_{n=0}^{N-1} (1) = N$$

6.7 (i) N.L., S.I., Memoryless, Causal

(ii) N.L., S.I., Memory, Causal

(iii) L., S.V., memoryless, Causal

(iv) N.L., S.V, memoryless, Causal

(v) L., S.I., memory, Causal

(vi) L., S.I., memory, Causal

(vii) Linear, Memory, noncausal

Let $x_1(n) = x(n-N)$, then $y_1(n) = \sum_{k=0}^{\infty} x_1(k) = \sum_{k=0}^{\infty} x(k-N)$

$$= \sum_{k=-N}^{\infty} x(k)$$

$$y(n-N) = \sum_{k=0}^{\infty} x(k)$$

$\therefore y_1(n) \neq y(n-N) \quad \therefore$ Not S.I

(viii) $y(n) = \sum_{k=0}^{n} x(n) = (n+1)\,x(n)$

L, S.V., memoryless

(ix) $y(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$

L, S.I., Memory, Causal

(x) L, S.I. Memory, Noncausal

(xi) Nonlinear, memory, noncausal, S.I.

(xii) Linear, memoryless, Causal, S.V.

(xiii) Nonlinear, S.I., memoryless, Causal

6.8 (i)
$$y(n) = \sum_{k=-\infty}^{n} x(k) = \begin{cases} 2(-n-6) & -5 \le n \le -1 \\ 2(n-4) & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(ii)
$$y(n) = (\frac{1}{2})^n u(n) + (\frac{1}{2})^{n-1} u(n-1) + (\frac{1}{2})^n u(n) * (\frac{1}{3})^n u(n)$$
$$= (\frac{1}{2})^n u(n) + (\frac{1}{2})^{n-1} u(n-1) + 6[(\frac{1}{2})^{n+1} - (\frac{1}{3})^{n+1}]u(n)$$
$$= 4(\frac{1}{2})^n u(n) + (\frac{1}{2})^{n-1} u(n-1) - 2(\frac{1}{3})^n u(n)$$

(iii)
$$y(n) = \sum_{k=0}^{n} h(n) = \begin{cases} n+1 & 0 \le n \le 9 \\ 10 & n \ge 10 \end{cases}$$

(iv)
$$y(n) = (\frac{1}{3})^n u(n) + (\frac{1}{3})^n * (\frac{1}{3})^n u(n)$$
$$= (\frac{1}{3})^n u(n) + (n+1)(\frac{1}{3})^n$$

(v)
$$y(n) = (\frac{1}{3})^n u(n) + (\frac{1}{3})^n * (\frac{1}{2})^n u(n)$$
$$= (\frac{1}{3})^n u(n) + 3(\frac{1}{2})^n u(n) - 2(\frac{1}{3})^n$$
$$= [3(\frac{1}{2})^n u(n) - 2(\frac{1}{3})^n]u(n)$$

(vi)

$$y(n) = \begin{cases} \sum\limits_{k=0}^{n} k & 0 \le n \le 9 \\ \sum\limits_{k=n-9}^{n} k & n \ge 10 \end{cases}$$

$$= \begin{cases} \frac{1}{2} n(n+1) & 0 \le n \le 9 \\ 10n - 4 & n \ge 10 \end{cases}$$

6.9 (i)

n	-2	-1	0	1	2	3	4	5	
$x(n)$	1	$-1/2$	$1/4$	$-1/8$	$1/16$				$\times 1$
$x(n-1)$		1	$-1/2$	$1/4$	$-1/8$	$1/16$			$\times -1$
$x(n-2)$			1	$-1/2$	$1/4$	$-1/8$	$1/16$		$\times 1$
$x(n-3)$				1	$-1/2$	$1/4$	$-1/8$	$1/16$	$\times -1$

$$y(n) = \left\{ 1, -\frac{3}{2}, \underset{\uparrow}{\frac{7}{4}}, -\frac{15}{8}, \frac{15}{16}, -\frac{7}{16}, \frac{3}{16}, -\frac{1}{16} \right\}$$

(ii)

	n	0	1	2	3	4	5	6	7	8
$2\times$	$x(n)$	1	2	3	0	-1				
$-1\times$	$x(n-1)$		1	2	3	0	-1			
$3\times$	$x(n-2)$			1	2	3	0	-1		
$1\times$	$x(n-3)$				1	2	3	0	-1	
$-2\times$	$x(n-4)$					1	2	3	0	-1

$$y(n) = \{2, 3, 7, 4, 7, 0, -9, -1, 2\}$$

(iii)

	n	-2	-1	0	1	2	3	4	5
$2\times$	$x(n)$	3	$1/2$	$-1/4$	1	4			
$-1\times$	$x(n-1)$		3	$1/2$	$-1/4$	1	4		
$1/2\times$	$x(n-2)$			3	$1/2$	$-1/4$	1	4	
$-1/2\times$	$x(n-3)$				3	$1/2$	$-1/4$	1	4

$$y(n) = \left\{6, -2, \frac{1}{2}, 1, 6\frac{5}{8}, -3\frac{3}{8}, \frac{3}{2}, -2\right\}$$

(iv)

n	0	1	2	3	4	5	6	7	8
$x(n)$	-1	$1/2$	$3/4$	$-1/5$	1				
$x(n-1)$		-1	$1/2$	$3/4$	$-1/5$	1			
$x(n-2)$			-1	$1/2$	$3/4$	$-1/5$	1		
$x(n-3)$				-1	$1/2$	$3/4$	$-1/5$	1	
$x(n-4)$					-1	$1/2$	$3/4$	$-1/5$	1

$$y(n) = \{-1,\ -1/2,\ 1/4,\ 1/20,\ 21/20,\ 41/20,\ 31/20,\ 4/5,\ 1\}$$

6.10 (a)

$$h(n) = [h_1(n) * h_2(n) - h_4(n)] * h_3(n)$$

$$= [(n+1)(\frac{1}{3})^n u(n) - (\frac{1}{2})^n u(n)] * u(n)$$

$$= \sum_{k=0}^{n}(k+1)(\frac{1}{3})^k - \sum_{k=0}^{n}(\frac{1}{2})^k$$

$$= -\frac{1}{2}n(\frac{1}{3})^n u(n) - \frac{5}{4}(\frac{1}{3})^n u(n) + \frac{9}{4}u(n) - 2u(n) + (\frac{1}{2})^n u(n)$$

$$= -\frac{1}{2}n(\frac{1}{3})^n u(n) - \frac{5}{4}(\frac{1}{3})^n u(n) + \frac{1}{4}u(n) + (\frac{1}{2})^n u(n)$$

(b)

$$s(n) = \sum_{k=0}^{n} h(k) = \sum_{k=0}^{n}[-\frac{1}{2}k(\frac{1}{3})^k - \frac{5}{4}(\frac{1}{3})^k + \frac{1}{4} + (\frac{1}{2})^k]$$

$$= [-\frac{1}{6}(\frac{1}{3})^n + \frac{5}{24}(\frac{1}{3})^n - (\frac{1}{2})^n + \frac{1}{4}n + \frac{9}{8}]u(n)$$

6.11 (a)

$$h(n) = [(\frac{1}{2})^n u(n) * \delta(n) - (\frac{1}{3})^n u(n)] * (\frac{1}{3})^n u(n)$$

$$= (\frac{1}{2})^n u(n) * (\frac{1}{3})^n u(n) - (\frac{1}{3})^n u(n) * (\frac{1}{3})^n u(n)$$

$$= 6[(\frac{1}{2})^{n+1} - (\frac{1}{3})^{n+1}]u(n) - (n+1)(\frac{1}{3})^n u(n)$$

$$= 3(\frac{1}{2})^n u(n) - 3(\frac{1}{3})^n u(n) - n(\frac{1}{3})^n u(n)$$

(b)

$$s(n) = \sum_{k=0}^{n} h(k) = \sum_{k=0}^{n} 3(\frac{1}{2})^k - 3(\frac{1}{3})^k - k(\frac{1}{3})^k$$

$$= \left[\frac{3}{4} - 3(\frac{1}{2})^n + \frac{9}{4}(\frac{1}{3})^n \frac{+n}{2}(\frac{1}{3})^n\right]u(n)$$

6.12 $$\sum_{k=n_0}^{N-1+n_0} x_1(k)\,x_2(n-k) = \sum_{k=n_0}^{-1} x_1(k)\,x_2(n-k) + \sum_{k=0}^{N-1+n_0} x_1(k)\,x_2(n-k)$$

$$= \sum_{m=n_0+N}^{N-1} x_1(m-N)\,x_2(n-m+N) + \sum_{K=0}^{N-1+n_0} x_1(k)\,x_2(n-k)$$

$$= \sum_{k=n_0+N}^{N-1} x_1(k)\,x_2(n-k) + \sum_{K=0}^{N-1+n_0} x_1(k)\,x_2(n-k) = \sum_{k=0}^{N-1} x_1(k)\,x_2(n-k)$$

6.13 (a) Do a circular shift of $x(n)$ so that the origin is at the left, to get

$$x(n) = \{\frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, 1, -\frac{1}{2}\}$$

Zero-pad $h(n)$ to get $h(n) = \{1, -1, 1, -1, 0\}$

Form the table

n	0	1	2	3	4	
$h(n)$	1	-1	1	-1	0	$\times\frac{1}{4}$
$h(n\text{-}1)$	0	1	-1	1	-1	$\times -\frac{1}{8}$
$h(n\text{-}2)$	-1	0	1	-1	1	$\times\frac{1}{16}$
$h(n\text{-}2)$	1	-1	0	1	-1	$\times 1$
$h(n\text{-}3)$	-1	1	-1	0	1	$\times -\frac{1}{2}$

$$y_p(n) = \{\frac{27}{16}, -\frac{15}{8}, \frac{15}{16}, \frac{9}{16}, -\frac{21}{16}\}$$

and is $y_l(n)$ of Problem 6.9(a) with the last three points aliased into the first 3 and the result circularly shifted by 2 points.

(b)

n	0	1	2	3	4	
$h(n)$	2	-1	3	1	-2	x1
$h(n-1)$	-2	2	-1	3	1	x2
$h(n-2)$	1	-2	2	-1	3	x3
$h(n-2)$	3	1	-2	2	-1	x0
$h(n-3)$	-1	3	1	-2	2	x−1

$$y_p(n) = \{2,\ -6,\ 6,\ 6,\ 7\}$$

and is $y_l(n)$ of Problem 6.9(b) with the last four points aliased into the first 4.

(c) Do a circular shift of $x(n)$ to get

$$x(n) = \{-\frac{1}{4}, 1, 4, 1, 3, \frac{1}{2}\}$$

Zero-pad $h(n)$ to get $h(n) = \{2, -1, \frac{1}{2}, -\frac{1}{2}, 0\}$

Form the table

n	0	1	2	3	4	
$h(n)$	2	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\times -\frac{1}{4}$
$h(n-1)$	0	2	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\times -\frac{1}{8}$
$h(n-2)$	$-\frac{1}{2}$	0	2	-1	$\frac{1}{2}$	$\times \frac{1}{16}$
$h(n-2)$	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	-1	x1
$h(n-3)$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	$\times -\frac{1}{2}$

$$y_p(n) = \{-\frac{3}{2},\ 1,\ \frac{13}{8},\ \frac{21}{8},\ -\frac{1}{2}\}$$

and is $y_l(n)$ of Problem 6.9(c) with the last three points aliased into the first 3 and the result circularly shifted left by 2 points.

(d)

n	0	1	2	3	4	
$h(n)$	1	1	1	1	1	x-1
$h(n-1)$	1	1	1	1	1	x$\frac{1}{2}$
$h(n-2)$	1	1	1	1	1	x$\frac{3}{4}$
$h(n-2)$	1	1	1	1	1	x$\frac{-1}{5}$
$h(n-3)$	1	1	1	1	1	x1

$$y_p(n) = \{\frac{21}{20}, \frac{21}{20}, \frac{21}{20}, \frac{21}{20}, \frac{21}{20}\}$$

and is $y_l(n)$ of Problem 6.9(d) with the last three points aliased into the first 3.

6.14(a) $$y(nT) + 2\,\frac{y(nT - y[(n-1)T]}{T} = x(nT)$$

$$\left(1 + \frac{2}{T}\right)y(nT) - \frac{2}{T}\,y[(n-1)T] = x(nT)$$

(b) $$y(nT) + 2\,\frac{y[(n+1)T] - y(nT)}{T} = x(nT)$$

$$\frac{2}{T}\,y[(n+1)T] + \left(1 - \frac{2}{T}\right)y(nT) = x(nT)$$

6.15/6.16

The tables below give selected values of the true value of the integrated y(t), and the approximations using the backward difference $y_B(nT)$, the forward difference, $y_F(nT)$ and the trapezoidal approximation, $y_T(nT)$, computed using $T = 0.002$ and $T = 0.001$

$T = 0.002$

t	y(t)	$y_B(t)$	$y_F(t)$	$y_T(t)$
0	0	0	0	0
0.01	0.01	0.01	0.01	0.09
0.05	0.05	0.05	0.05	0.05
0.1	0.1	0.1	0.1	0.1
0.25	0.25	0.25	0.25	0.25
0.5	0.5	0.5	0.5	0.5
0.75	0.75	0.75	0.75	0.75
1	1	1	1	1
1.01	1.00995	1.0	1.00994	1.00895
1.1	1.095	1.0949	1.0949	1.094
1.2	1.18	1.1798	1.1798	1.179
1.5	1.375	1.3745	1.3745	1.374
1.6	1.42	1.4194	1.4194	1.419
1.7	1.455	1.4543	1.4543	1.454
1.8	1.48	1.4792	1.4792	1.479
1.9	1.495	1.4941	1.4941	1.495
2.0	1.5	1.499	1.499	1.499
3.0	1.5	1.499	1.499	1.499

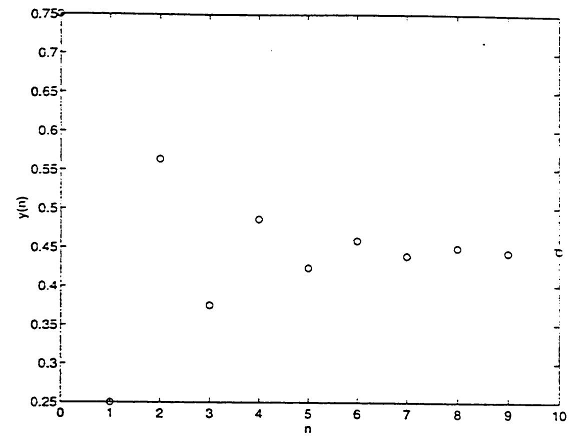
$T = 0.001$

t	y(t)	$y_B(t)$	$y_F(t)$	$y_T(t)$
0	0	0	0	0
0.01	0.01	0.01	0.01	0.01
0.05	0.05	0.05	0.05	0.05
0.9	0.9	0.9	0.9	0.9
1	1	1	1	1
1.01	1.00995	1.009945	1.009946	1.00995
1.1	1.095	1.09495	1.094951	1.095
1.2	1.18	1.1799	1.179901	1.179999
1.5	1.375	1.37475	1.374751	1.374999
1.6	1.42	1.4197	1.419701	1.419999
1.7	1.455	1.45465	1.454651	1.454999
1.8	1.48	1.4796	1.479601	1.47999
1.9	1.495	1.49455	1.494551	1.494999
2.0	1.5	1.4995	1.4995	1.499999
3.0	1.5	1.4995	1.4995	1.49999

It can be seen that all three methods provide good approximations to the integral in this example. The approximations get better as T becomes smaller.

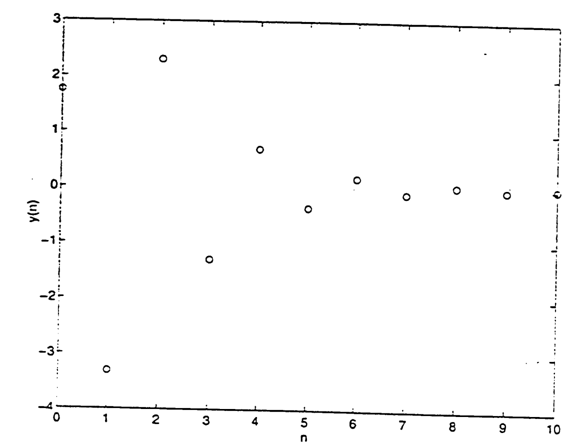
6.17 (a) $y(n) = -y(n-1) - \frac{1}{4}y(n-2) + 1$

$$y(0) = -y(-1) - \frac{1}{4}y(-2) + 1 = 0 - 0.25 + 1 = 0.75$$
$$y(1) = -y(0) - \frac{1}{4}y(-1) + 1 = -0.75 + 1 = 0.25$$
$$y(2) = -y(1) - \frac{1}{4}y(0) + 1 = 0.5625$$
$$y(3) = -y(2) - \frac{1}{4}y(1) + 1 = 0.375$$
$$y(4) = -y(3) - \frac{1}{4}y(2) + 1 = 0.4844$$
$$y(5) = -y(4) - \frac{1}{4}y(3) + 1 = 0.4219$$



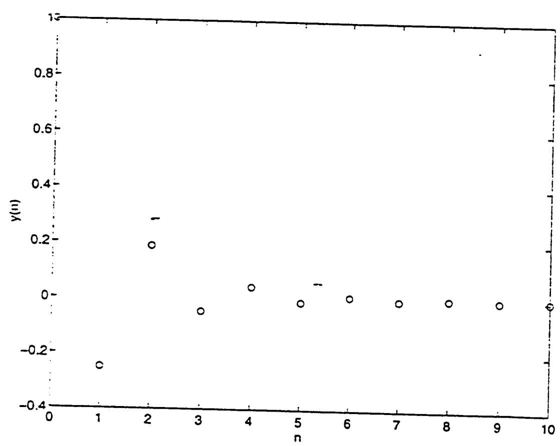
(b) $y(n) = -\frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + (\frac{1}{3})^n$

$$y(0) = -\frac{3}{4}y(-1) - \frac{1}{8}y(-2) + 1 = 1.75$$
$$y(1) = -\frac{3}{4}y(0) - \frac{1}{8}y(-1) + \frac{1}{3} = -3.3125$$
$$y(2) = -\frac{3}{4}y(1) - \frac{1}{8}y(0) + \frac{1}{9} = 2.3027$$
$$y(3) = -\frac{3}{4}y(2) - \frac{1}{8}y(1) + \frac{1}{27} = -1.3006$$
$$y(4) = -\frac{3}{4}y(3) - \frac{1}{8}y(2) + \frac{1}{81} = 0.6917$$
$$y(5) = -\frac{3}{4}y(4) - \frac{1}{8}y(3) + \frac{1}{243} = -0.3548$$



(c) $y(n) = -\frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + (\frac{1}{2})^n$

$$y(0) = -\frac{3}{4}y(-1) - \frac{1}{8}y(-2) + 1 = 1$$
$$y(1) = -\frac{3}{4}y(0) - \frac{1}{8}y(-1) + \frac{1}{2} = -0.25$$
$$y(2) = -\frac{3}{4}y(1) - \frac{1}{8}y(0) + \frac{1}{4} = 0.1875$$
$$y(3) = -\frac{3}{4}y(2) - \frac{1}{8}y(1) + \frac{1}{8} = -0.0469$$
$$y(4) = -\frac{3}{4}y(3) - \frac{1}{8}y(2) + \frac{1}{16} = 0.043$$
$$y(5) = -\frac{3}{4}y(4) - \frac{1}{8}y(3) + \frac{1}{32} = -0.0107$$



(d) $y(n+1) = -\frac{1}{2}\,y(n-1) + x(n) - \frac{1}{2}\,x(n-1)$

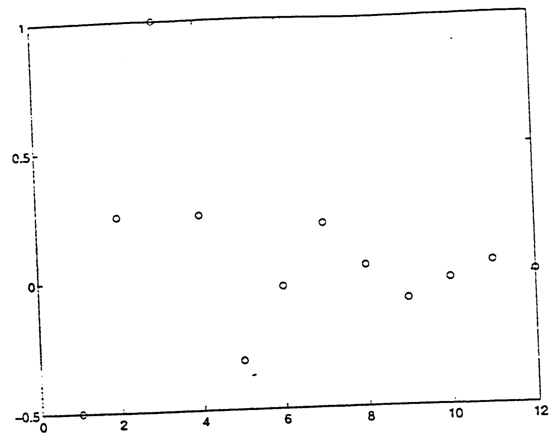
$$y(1) = -\frac{1}{2}\,y(0) + \frac{1}{4} - \frac{1}{2} = -0.5$$

$$y(2) = -\frac{1}{2}\,y(1) + \frac{1}{8} - \frac{1}{4} = 0.25$$

$$y(3) = -\frac{1}{2}\,y(2) + \frac{1}{16} - \frac{1}{8} = 1$$

$$y(4) = -\frac{1}{2}\,y(3) + \frac{1}{32} - \frac{1}{16} = -0.3125$$

$$y(5) = -\frac{1}{2}\,y(4) + \frac{1}{64} - \frac{1}{32} = -0.0312$$



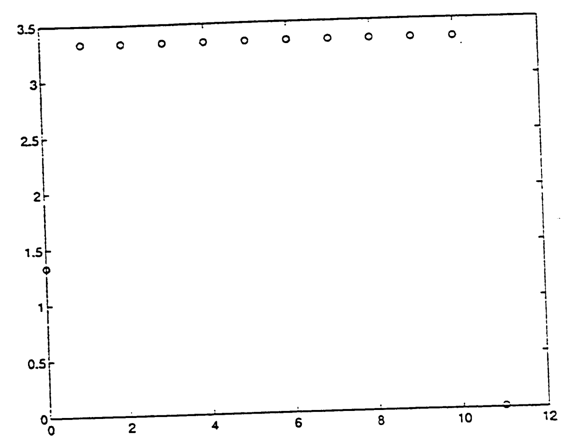
(e) $y(n) = x(n) + \frac{1}{3}\,x(n-1) + 2x(n-2)$

$$y(0) = x(0 = 1$$

$$y(1) = 1 + \frac{1}{3} = 1\,\frac{1}{3}$$

$$y(2) = 1 + \frac{1}{3} + 2 = 3\,\frac{1}{3}$$

$$y(3) = y(4) = \ldots = y(2) = 3\,\frac{1}{3}$$



6.18 (i) Characteristic equation is $z^2 - z + \frac{1}{2} = 0$ with roots

$$z = \frac{1}{2} \pm j\,\frac{1}{2} = e^{\pm j\frac{\pi}{4}} \text{ so that } y_h(n) = A_1 e^{j\frac{\pi}{4}n} + A_2 e^{-j\frac{\pi}{4}n}$$

Substituting the initial conditions and solving for A_1 and A_2 gives

$$y_h(n) = e^{j\frac{\pi}{4}(n+3)} + e^{-j\frac{\pi}{4}(n+3)} = 2\cos(\frac{\pi}{4}\,(n+3))u(n)$$

(ii) Characteristic equation is $z^2 - \frac{1}{6}\,z - \frac{1}{6} = 0$ with roots

$$z = -\frac{1}{3}\,,\ \frac{1}{2} \text{ so that } y_h(n) = A_1(-\frac{1}{3})^n + A_2(\frac{1}{2})^n$$

Solving for A_1 and A_2 gives

$$y_h(n) = \frac{1}{30}\,(-\frac{1}{3})^n u(n) + \frac{1}{10}\,(\frac{1}{2})^n u(n)$$

(iii) Characteristic equation is $z^2 - z + \frac{1}{4} = 0$ with roots

$z = \frac{1}{2}, \frac{1}{2}$ so that $y_h(n) = A_1(\frac{1}{2})^n + A_2 n(\frac{1}{2})^n$

Solving for A_1 and A_2 gives

$$y_h(n) = \frac{3}{4}(-\frac{1}{2})^n u(n) - \frac{1}{4}n(-\frac{1}{2})^n u(n)$$

(iv) Characteristic equation is $z^2 - \frac{3}{2}z + \frac{1}{2} = 0$ with roots

$z = 1, \frac{1}{2}$ so that $y_h(n) = A_1 + A_2(\frac{1}{2})^n$.

Solving for A_1 and A_2 gives

$$y_h(n) = 2u(n) - \frac{1}{2}(\frac{1}{2})^n u(n)$$

(v) Characteristic equation is $z^2 - \frac{1}{4}z - \frac{1}{8} = 0$ with roots

$z = -\frac{1}{4}, \frac{1}{2}$ so that $y_h(n) = A_1(-\frac{1}{4})^n + A_2(\frac{1}{2})^n$

Solving for A_1 and A_2 gives

$$y_h(n) = -\frac{1}{4}(-\frac{1}{4})^n u(n) + \frac{1}{4}(\frac{1}{2})^n u(n)$$

6.19 (i) Ch. Eqn. is $z^2 + \frac{1}{6}z - \frac{1}{6} = 0$, Roots are $\frac{1}{3}, -\frac{1}{2}$

$y_h(n) = A\left(\frac{1}{3}\right)^n + B\left(-\frac{1}{2}\right)^n$

Let $y_p(n) = k_1 \cos\frac{3n}{4} + k_2 \sin\frac{3n}{4}$

$$y_p(n-1) = k_1 \cos\frac{3(n-1)}{4} + k_2 \sin\frac{3(n-1)}{4}$$

$$= k_1\left[\cos\frac{3\pi n}{4}\cos\frac{3\pi}{4} + \sin\frac{3\pi n}{4}\sin\frac{3\pi}{4}\right]$$

$$+ k_2\left[\sin\frac{3\pi n}{4}\cos\frac{3\pi}{4} - \cos\frac{3\pi n}{4}\sin\frac{3\pi}{4}\right]$$

$$y_p(n-2) = k_1\left[\cos\frac{3\pi n}{4}\cos\frac{3\pi}{2} + \sin\frac{3\pi n}{4}\sin\frac{3\pi}{2}\right]$$

$$+\,k_2\left[\cos\frac{3\pi}{2}\sin\frac{3\pi n}{4} - \cos\frac{3\pi n}{4}\sin\frac{3\pi}{2}\right]$$

Use $\cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} = -\sin\frac{3\pi}{4}$, $\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$

and substitute in d.e. to get

$$k_1 - \frac{k_1}{6\sqrt{2}} - \frac{k_2}{6\sqrt{2}} - \frac{k_2}{6} = 2 - \frac{1}{\sqrt{2}}$$

$$k_2 + \frac{k_1}{6\sqrt{2}} - \frac{k_2}{6\sqrt{2}} + \frac{k_1}{6} = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{38-5\sqrt{2}}\begin{bmatrix} 78 - 21\sqrt{2} \\ -12 + 9\sqrt{2} \end{bmatrix}$$

$$\therefore\quad y(n) = A_1\left(-\tfrac{1}{2}\right)^n + A_2\left(\tfrac{1}{3}\right)^n + k_1\cos\frac{3\pi n}{4} + k_2\sin\frac{3\pi n}{4}$$

$$\left.\begin{aligned} n=-1 \text{ gives } & -5A - 3B + \tfrac{15}{16} = 1 \\ n=-2 \text{ gives } & 25A + 9B + \tfrac{45}{16} = 1 \end{aligned}\right\}\quad \begin{aligned} A &= -\tfrac{13}{80} \\ B &= 1/4 \end{aligned}$$

(v) $y_p(n) = k\left(\frac{1}{4}\right)^n$

Substitute in d.e to get $k = \frac{-51}{2}$

$$y(n) = A\left(-\tfrac{1}{4}\right)^n + B\left(\tfrac{1}{3}\right)^n - \tfrac{51}{2}\left(\tfrac{1}{4}\right)^n$$

$y(0) = y(1) = 0$ gives $A = \frac{13}{2}$, $B = \frac{153}{-7}$

6.20 (i) For $n \geq 1$, $y(n) - y(n-1) + \frac{1}{2}y(n-2) = 0$

so that $h(n) = A_1 e^{-j\frac{\pi}{4}n} + A_2 e^{j\frac{\pi}{4}n}$.

With $y(-1) = 0$, $y(0) = y(-1) - \frac{1}{2}y(-2) + 1 = 1$, solving for the constants gives $A_1 = A_2{}^* = \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}$

so that

$$h(n) = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}(n-1)} \frac{+1}{\sqrt{2}} e^{j\frac{\pi}{4}(n-1)} = \sqrt{2}\cos(\frac{\pi}{4}(n-1))u(n)$$

(ii) For $n \geq 2$, $y(n) - \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) = 0$

so that $h(n) = A_1(-\frac{1}{3})^n + A_2(\frac{1}{2})^n$.

With $y(0) = 1$, we get $y(1) = \frac{1}{6} y(0) - \frac{1}{2} x(0) = -\frac{1}{3}$.

Solving for the constants gives $A_1 = -\frac{1}{5}, A_2 = \frac{6}{5}$

so that

$$h(n) = -\frac{1}{5}(-\frac{1}{3})^n u(n) + \frac{6}{5}(\frac{1}{2})^n u(n)$$

(iii) For $n \geq 1$, $y(n) - y(n-1) + \frac{1}{4} y(n-2) = 0$

so that $h(n) = A_1(\frac{1}{2})^n + A_2 n(\frac{1}{2})^n$.

With $y(-1) = 0$, we have $y(0) = x(0) = 1$. Solving for the constants gives $A_1 = A_2 = 1$, so that

$$h(n) = (n+1)(\frac{1}{2})^n u(n)$$

(iv) For $n \geq 1$, $y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = 0$

so that $h(n) = A_1 + A_2(\frac{1}{2})^n$.

6.21 (i) $y(n) = A\left(\frac{1}{3}\right)^n + B\left(-\frac{1}{2}\right)^n$ for $n \geq 2$

$\left.\begin{array}{l} y(0) = 1 \\ y(1) = \frac{1}{3} \end{array}\right\}$ $h(n) = \left(\frac{1}{3}\right)^n u(n)$ Since $A = 1,\ B = 0$

(ii) $y_n(n) = A\left(-\frac{1}{2}\right)^n$ for $n \geq 1$

$y(0) = 1 \quad \therefore \quad h(n) = \left(-\frac{1}{2}\right)^n u(n)$

(iii) Same as (ii)

(iv) $y_n(n) = A\left(-\frac{1}{5}\right)^n + B\left(-\frac{1}{3}\right)^n$ for $n \geq 1$

$\left.\begin{array}{l} y(0) = 1 \\ y(-1) = 0 \end{array}\right\}$ $\therefore A = \frac{3}{2}$, $B = \frac{5}{2}$ $\quad \therefore h(n) = \frac{3}{2}\left(-\frac{1}{5}\right)^n + \frac{5}{2}\left(-\frac{1}{3}\right)^n$

for $n \geq 0$

(v) Same as 6.18 (v)

6.22 Ch. eqn. of the assumed eq^n is $z^2+az+b=0$

From given $h(n)$, roots of ch. eqn. must be $\frac{1}{2}, -\frac{1}{2}$

$\therefore \ (z-\frac{1}{2})(z+\frac{1}{2}) = z^2+az+b$, $\quad a=0$, $b=-\frac{1}{4}$

For $n \geq 2$, eq^n is $y(n) - \frac{1}{4}\, y(n-2) = 0$

with solution $y(n) = A(\frac{1}{2})^n + B(-\frac{1}{2})^n$

I.C. are $y(0) = c$, $y(1) = d$

Since $A = B = 1$, $c = 2$, $d = 0$

$\therefore$ Eqn is $y(n) - \frac{1}{4}\, y(n-2) = 2\, x(n)$

6.23 Let $h_1(n) = (-\frac{1}{2})^n u(n)$

Then $h(n) = h_1(n) + \frac{1}{2} h_1(n-1)$

clearly $h_1(n)$ is impulse response of a system of the form

$$y(n) + \tfrac{1}{2}\, y(n-1) = x(n)$$

$\therefore$ $h(n)$ corresponds to $y(n) + \frac{1}{2}\, y(n-1) = x(n) + \frac{1}{2}\, x(n-1)$

6.24(a) For $n \geq 1$, we have $\sum_{k=0}^{N} a_k\, y(n-k) = 0$

$$y(0) = \frac{1}{a_0}\left[x(0) - \sum_{k=1}^{N} a_k\, y(-k)\right] = \frac{1}{a_0}$$

(b) Let $y(n) - \frac{3}{4}\, y(n-1) + \frac{1}{8}\, y(n-2) = x(n)$ have impulse response $h^*(n)$

Then $h^*(n) = A(\frac{1}{2})^n - B(\frac{1}{4})^n$

with $\left.\begin{array}{l} h^*(0) = 1 \\ h^*(-1) = 0 \end{array}\right\}$ Gives $A = 2$, $B = 1$

$\therefore\ h^*(n) = \left[2(\frac{1}{2})^n - (\frac{1}{4})^n\right] u(n)$

$\therefore\ h(n) = \left[2(\frac{1}{2})^n - (\frac{1}{4})^n\right] u(n) + \frac{1}{2}\left[2(\frac{1}{2})^{n-1} - (\frac{1}{4})^{n-1}\right] u(n-1)$

$$= \begin{cases} 1 & n=0 \\ 4\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{4}\right)^n & n \geq 1 \end{cases} = \left[4\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{4}\right)^n\right]u(n)$$

(c) (i) Need to solve $y(n) + y(n-1) + y(n-2) = 0$

with $y(-1) = 0$, $y(0) = 1$

Gives $y(n) = A \cos \frac{2\pi}{3} n + B \sin \frac{2\pi}{3} n$

$$\left.\begin{aligned} y(-1) = 0 &= \tfrac{1}{2}A - \tfrac{\sqrt{3}}{2}B \\ y(0) = 1 &= A \end{aligned}\right\} \begin{aligned} A &= 1 \\ B &= \tfrac{1}{\sqrt{3}} \end{aligned}$$

$\therefore\ h(n) = \left[\cos \frac{2\pi}{3} n + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3} n\right] u(n)$

(ii) To find $h(n)$, solve $y(n) - \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = 0$

for $n \geq 1$ with $y(-1) = 0$, $y(0) = 1$

$\therefore\ h(n) = \left[\frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{1}{3}\left(-\frac{1}{4}\right)^n\right] u(n)$

$\therefore\ h(n) = \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n\right] u(n) - \frac{1}{2}\left[\frac{2}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{3}\left(-\frac{1}{4}\right)^{n-1}\right] u(n-1)$

$= \left(-\frac{1}{4}\right)^n u(n)$

(iii) Solve $y(n) - y(n-1) + \frac{15}{64} y(n-2) = 0 \qquad n \geq 1$

$y(-1) = 0$, $y(0) = 1$

gives $h(n) = \left[\frac{5}{2}\left(\frac{5}{8}\right)^n - \frac{3}{2}\left(\frac{3}{8}\right)^n\right] u(n)$

(iv) Solve $y(n+2) + \frac{2}{3} y(n+1) + \frac{1}{9} y(n) = 0$ for $n \geq 1$

with $y(1) = 0$, $y(2) = 1$

Solution is $y(n) = A\left(-\frac{1}{3}\right)^n + Bn\left(-\frac{1}{3}\right)^n$

$$\left.\begin{aligned} n=1 \text{ gives } \quad & -\tfrac{1}{3}A - \tfrac{1}{3}B = 0 \\ n=2 \text{ gives } \quad & -\tfrac{1}{9}A + \tfrac{2}{9}B = 1 \end{aligned}\right\} \begin{aligned} A &= -9 \\ B &= 9 \end{aligned}$$

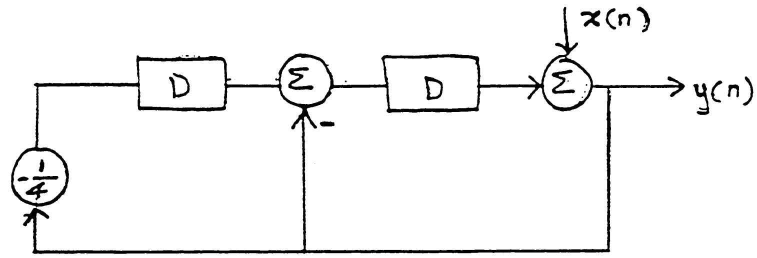
$\therefore\ h(n) = \left[-9\left(-\frac{1}{3}\right)^n + 9n\left(-\frac{1}{3}\right)^n\right] u(n)$

(v) For $y(n) = x(n)$, we have $h^*(n) = \delta(n)$

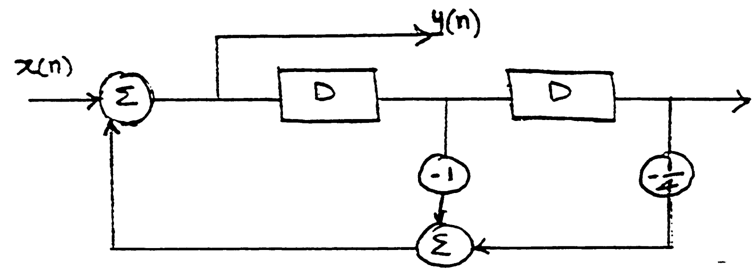
$$\therefore\ h(n) = h^*(n) - 3h^*(n-1) + 2h^*(n-2) - h^*(n-3)$$

$$= \delta(n) - 3\delta(n-1) + 2\delta(n-2) - \delta(n-3)$$

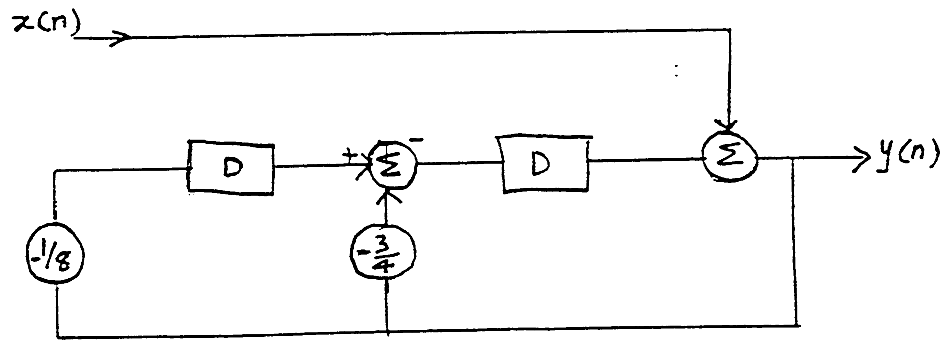
6.25 (i) (a)



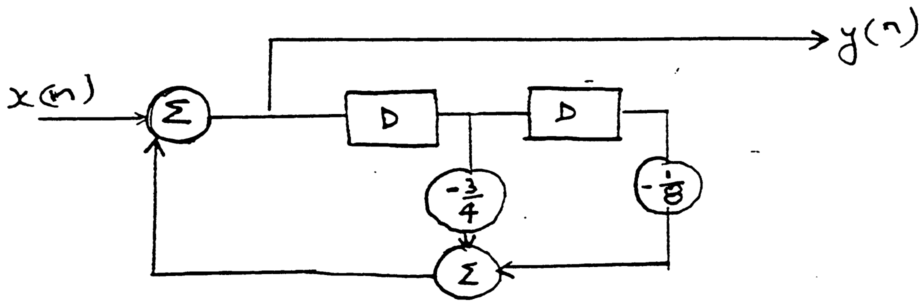
(b)



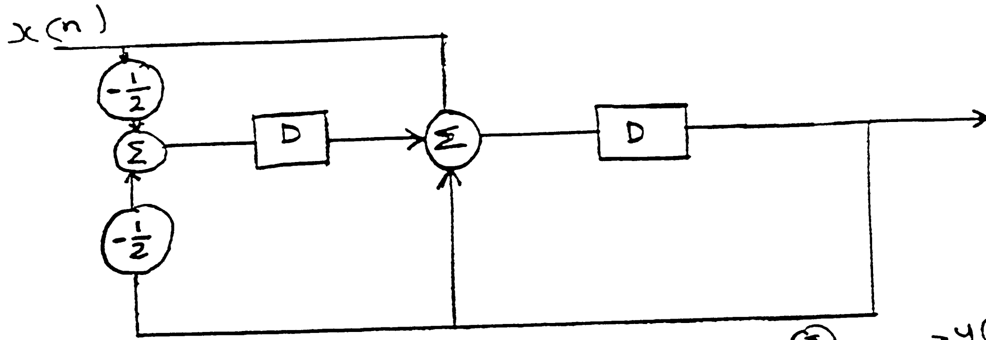
(ii)
(iii) (a)



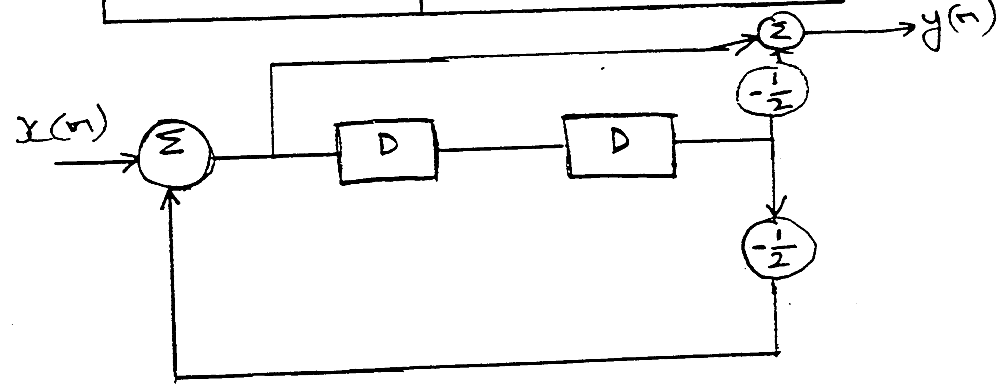
(b)



(iv) (a)

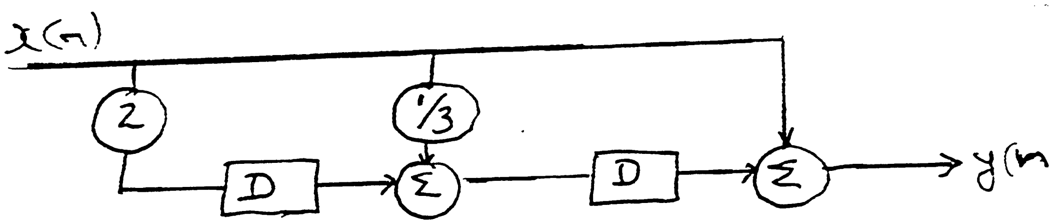

x(n)
$-\frac{1}{2}$
Σ
D
Σ
D
$-\frac{1}{2}$

(b)

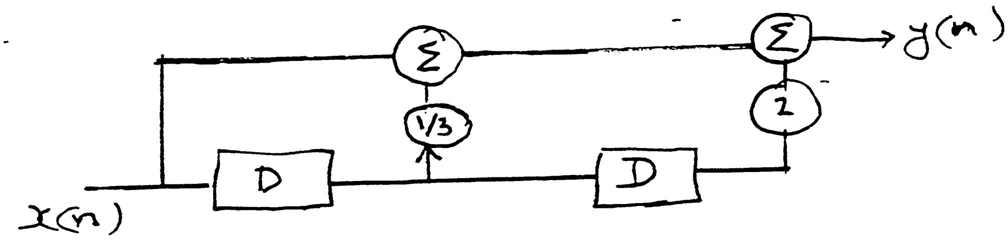

x(n)
Σ
D
D
Σ
y(n)
$-\frac{1}{2}$
$-\frac{1}{2}$

(v)

(a)


x(n)
2
1/3
D
Σ
D
Σ
y(n)

(b)


Σ
Σ
y(n)
1/3
2
D
D
x(n)

6.26 (i) (a)

$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) - y(n) = -v_1(n) + v_2(n) - x(n)$$

$$v_2(n+1) = -\frac{1}{4}y(n) = -\frac{1}{4}v_1(n) - \frac{1}{4}x(n)$$

(b)

$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{4}\hat{v}_1(n) - \hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{4}\hat{v}_1(n) - \hat{v}_2(n) + x(n)$$

(ii) (a)

(iii)

$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) - \frac{3}{4}y(n) = -\frac{3}{4}v_1(n) + v_2(n) - \frac{3}{4}x(n)$$

$$v_2(n+1) = -\frac{1}{8}y(n) = -\frac{1}{8}v_1(n) - \frac{1}{8}x(n)$$

(b)

$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{8}\hat{v}_1(n) - \frac{3}{4}\hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{8}\hat{v}_1(n) - \frac{3}{4}\hat{v}_2(n) + x(n)$$

(iv) (a)

$$y(n) = v_1(n)$$

$$v_1(n+1) = v_2(n) + x(n)$$

$$v_2(n+1) = -\frac{1}{2}v_1(n) - \frac{1}{2}x(n)$$

(b)

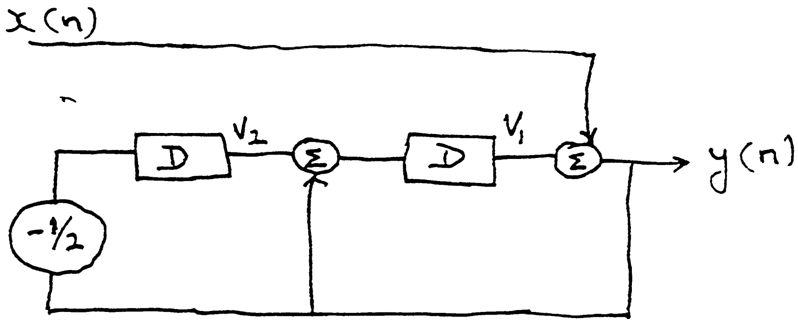
$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{2}\hat{v}_1(n) + x(n)$$

$$y(n) = -\frac{1}{2}\hat{v}_1(n) + \hat{v}_2(n)$$

(v) System has no states.

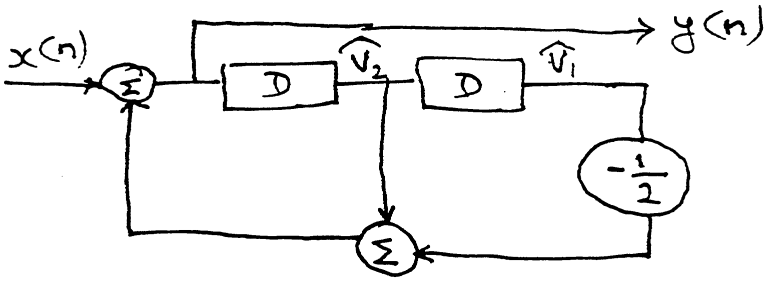
6.27 (i)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + y(n) = v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = -\frac{1}{2}y(n) = -\frac{1}{2}v_1(n) - \frac{1}{2}x(n)$$

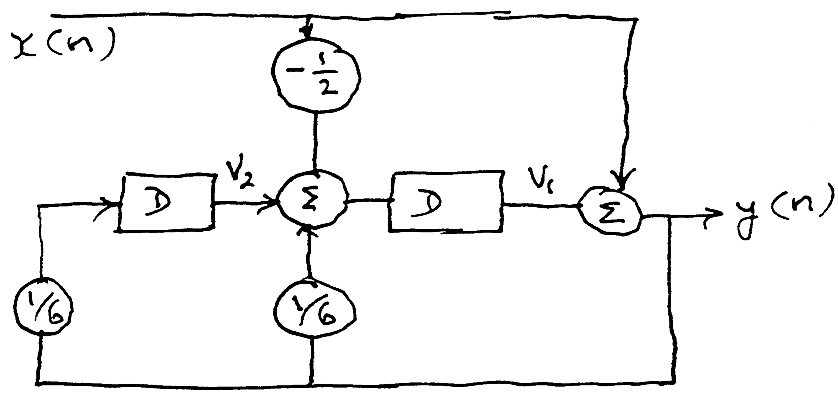


$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = x(n) - \frac{1}{2}\hat{v}_1(n) + \hat{v}_2(n)$$

$$y(n) = -\frac{1}{2}\hat{v}_1(n) + \hat{v}_2(n) + x(n)$$

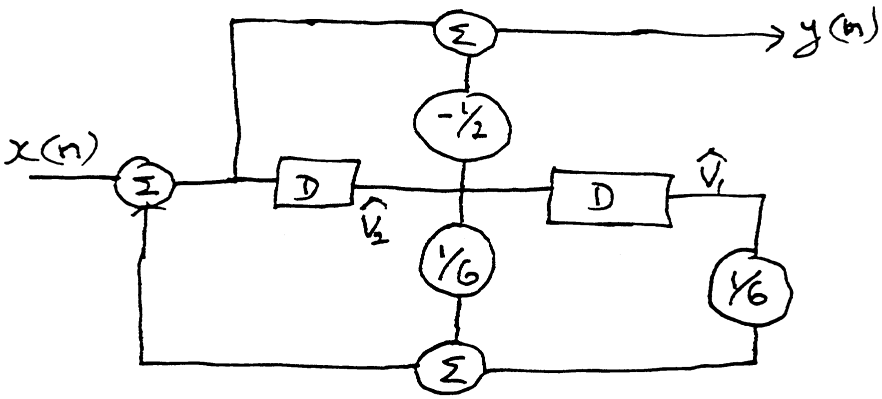
(ii)



$$y(n) = V_1(n) + x(n)$$

$$V_1(n+1) = V_2(n) + \frac{1}{6}y(n) - \frac{1}{2}x(n) = \frac{1}{6}V_1(n) + V_2(n) - \frac{1}{3}x(n)$$

$$V_2(n+1) = \frac{1}{6}y(n) = \frac{1}{6}V_1(n) + \frac{1}{6}x(n)$$



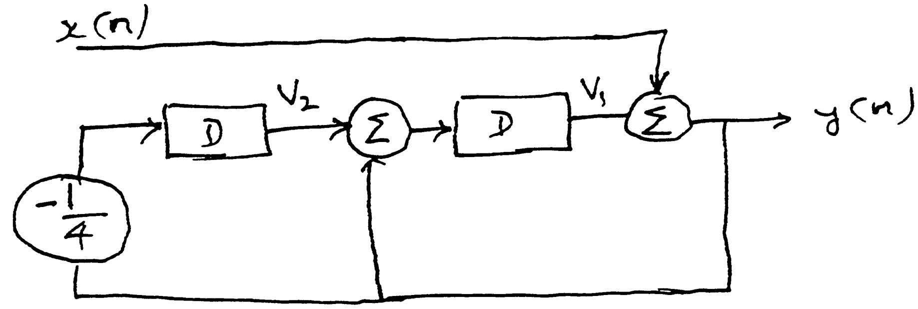
$$\hat{V}_1(n+1) = \hat{V}_2(n)$$

$$\hat{V}_2(n+1) = \frac{1}{6}\hat{V}_1(n) + \frac{1}{6}\hat{V}_2(n) + x(n)$$

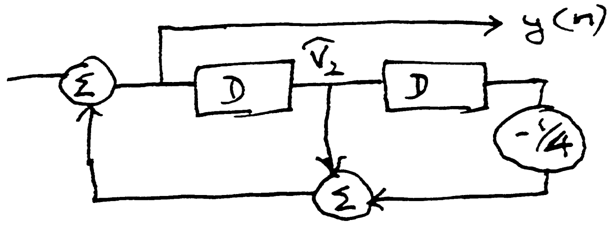
$$y(n) = -\frac{1}{2}\hat{V}_2(n) + \hat{V}_2(n+1)$$

$$= \frac{1}{6}\hat{V}_1(n) - \frac{1}{3}\hat{V}_2(n) + x(n)$$

(iii)

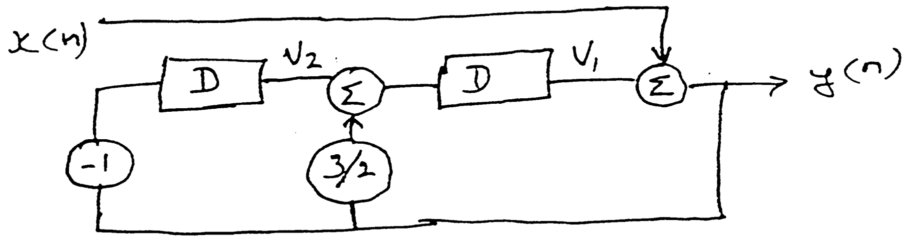


$$y(n) = V_1(n) + x(n)$$
$$V_1(n+1) = V_2(n) + y(n) = V_1(n) + V_2(n) + x(n)$$
$$V_2(n+1) = -\frac{1}{4}y(n) = -\frac{1}{4}V_1(n) - \frac{1}{4}x(n)$$



$$\hat{V}_1(n+1) = \hat{V}_2(n)$$
$$\hat{V}_2(n+1) = -\frac{1}{4}\hat{V}_1(n) + \hat{V}_2(n) + x(n)$$
$$y(n) = -\frac{1}{4}\hat{V}_1(n) + \hat{V}_2(n) + x(n)$$

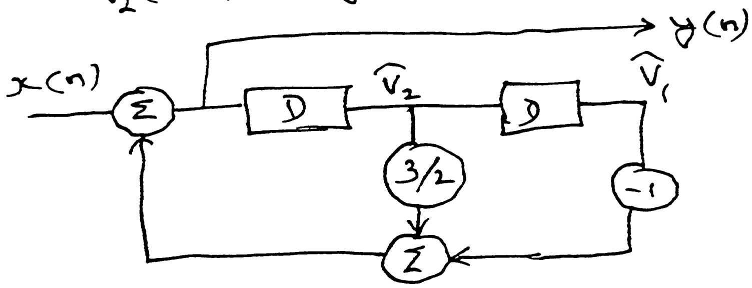
(iv)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + \frac{3}{2} y(n) = \frac{3}{2} v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = -y(n) = -v_1(n) - x(n)$$

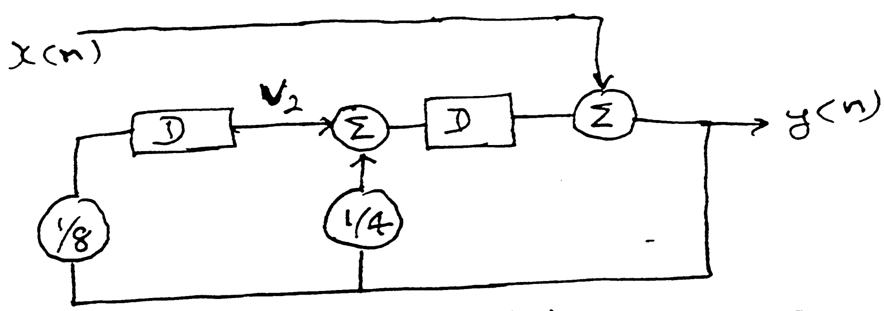


$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\hat{v}_1(n) + \frac{3}{2} \hat{v}_2(n) + x(n)$$

$$y(n) = -\hat{v}_1(n)$$

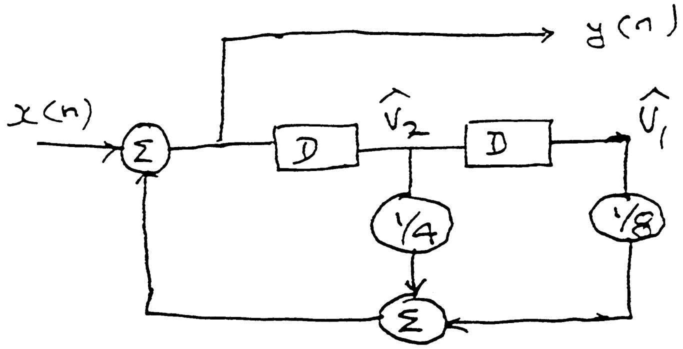
(v)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + \frac{1}{4} y(n) = \frac{1}{4} v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = \frac{1}{8} y(n) = \frac{1}{8} v_1(n) + \frac{1}{8} x(n)$$



$$\hat{V}_1(n+1) = \hat{V}_2(n)$$

$$\hat{V}_2(n+1) = \frac{1}{8}\hat{V}_1(n) + \frac{1}{4}\hat{V}_2(n) + x(n)$$

$$y(n) = \frac{1}{8}\hat{V}_1(n) + \frac{1}{4}\hat{V}_2(n) + x(n)$$

6.28 (i) Use Second Canonical form to get

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/24 & -9/24 & 13/12 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad c = [1 \ 0 \ 0] \quad d = 0$$

Characteristic equation is $z^2 - \frac{13}{12}z^2 + \frac{9}{24}z - \frac{1}{24} = 0$

with roots $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

$\therefore \ \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{4}\alpha_2 = \left(\frac{1}{2}\right)^n$

$\alpha_0 + \frac{1}{3}\alpha_1 + \frac{1}{9}\alpha_2 = \left(\frac{1}{3}\right)^n$

$$\alpha_0 + \tfrac{1}{4}\alpha_1 + \tfrac{1}{16}\alpha_2 = \left(\tfrac{1}{4}\right)^n$$

So that

$$\alpha_0 = 2\left(\tfrac{1}{2}\right)^n - 9\left(\tfrac{1}{3}\right)^n + 8\left(\tfrac{1}{4}\right)^n$$

$$\alpha_1 = -14\left(\tfrac{1}{2}\right)^n + 54\left(\tfrac{1}{3}\right)^n - 40\left(\tfrac{1}{4}\right)^n$$

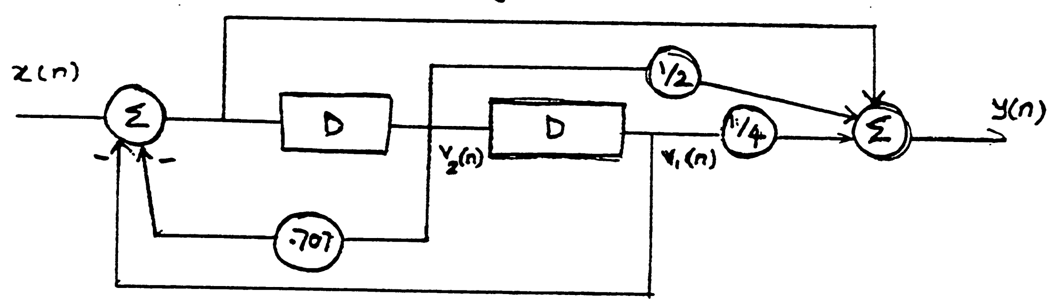
$$\alpha_2 = 24\left(\tfrac{1}{2}\right)^n - 72\left(\tfrac{1}{3}\right)^n + 48\left(\tfrac{1}{4}\right)^n$$

$$A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \alpha_0\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \alpha_1\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1/24 & -9/24 & 13/12\end{bmatrix} + \alpha_2\begin{bmatrix}0 & 0 & 1\\ 1/24 & -9/24 & 13/72\\ \frac{13}{288} & \frac{-35}{96} & \frac{115}{144}\end{bmatrix}$$

$$= \begin{bmatrix}2\left(\frac{1}{2}\right)^n - 9\left(\frac{1}{3}\right)^n + 8\left(\frac{1}{4}\right)^n & -14\left(\frac{1}{2}\right)^n + 54\left(\frac{1}{3}\right)^n - 40\left(\frac{1}{4}\right)^n & 24\left(\frac{1}{2}\right)^n - 72\left(\frac{1}{3}\right)^n + 48\left(\frac{1}{4}\right)^n\\ \left(\frac{1}{2}\right)^n - 3\left(\frac{1}{3}\right)^n - 4\left(\frac{1}{4}\right)^n & -7\left(\frac{1}{2}\right)^n + 18\left(\frac{1}{3}\right)^n - 10\left(\frac{1}{4}\right)^n & 12\left(\frac{1}{2}\right)^n - 24\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n\\ \frac{1}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n & -\frac{7}{2}\left(\frac{1}{2}\right)^n + 6\left(\frac{1}{3}\right)^n - \frac{5}{2}\left(\frac{1}{4}\right)^n & 6\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{4}\right)^n\end{bmatrix}$$

(ii) Second Canonical form gives



$$\therefore\quad v_1(n+1) = v_2(n)$$

$$v_2(n+1) = -v_1(n) - .707\, v_2(n)$$

$$y(n) = -.75\, v_1(n) - .207\, v_2(n) + x(n)$$

$$\therefore\ A = \begin{bmatrix}0 & 1\\ -.75 & -.707\end{bmatrix} \quad \underline{b} = \begin{bmatrix}0\\1\end{bmatrix} \quad \underline{c} = \begin{bmatrix}-.75 & -.207\end{bmatrix} \quad d = 1$$

Characteristic equation is $z^2 + \frac{1}{\sqrt{2}} z + 1 = 0$

with roots $z = e^{\pm j\theta}$ where $\theta = \tan^{-1}(-\sqrt{7})$

$$\therefore \quad \alpha_0 + \alpha_1 e^{j\theta} = e^{jn\theta}$$
$$\alpha_0 + \alpha e^{-j\theta} = e^{-jn\theta}$$

So that $\alpha_0 = \dfrac{-\sin(n-1)\theta}{\sin\theta}$ where $\sin\theta = \dfrac{1}{2\sqrt{2}}$

$$\alpha_1 = \frac{\sin n\theta}{\sin\theta}$$

$$\therefore \quad A^n = -\frac{\sin(n-1)\theta}{\sin\theta}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sin n\theta}{\sin\theta}\begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{\sin(n-1)\theta}{\sin\theta} & \dfrac{\sin n\theta}{\sin\theta} \\ -\dfrac{3}{4}\dfrac{\sin n\theta}{\sin\theta} & -\dfrac{\sin(n-1)\theta}{\sin\theta} - \dfrac{1}{\sqrt{2}}\dfrac{\sin n\theta}{\sin\theta} \end{bmatrix}$$

(iii) Second canonical form gives

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad d = 0$$

Characteristic eqn is $z^2 - 3z + 2 = 0$

with roots $z = +1, +2$

$\therefore \quad \alpha_0 + \alpha_1 = 1$ so that $\alpha_0 = 2 - 2^n$

$\alpha_0 + 2\alpha_1 = 2^n$ $\qquad \alpha_1 = 2^n - 1$

$$A^n = (2 - 2^n)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2^n - 1)\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2^n & 2^n - 1 \\ -2^{n+1} + 2 & -1 + 2^{n+1} \end{bmatrix}$$

6.29 $\underline{v}(n) = \sum_{j=0}^{n-1} A^{n-j-1} b$

(i) $$\underline{v}(n) = \sum_{k=0}^{n-1} \begin{bmatrix} 24\left(\frac{1}{2}\right)^k - 72\left(\frac{1}{3}\right)^k + 48\left(\frac{1}{4}\right)^k \\ 12\left(\frac{1}{2}\right)^k - 24\left(\frac{1}{3}\right)^k + 12\left(\frac{1}{4}\right)^k \\ 6\left(\frac{1}{2}\right)^k - 8\left(\frac{1}{3}\right)^k + 3\left(\frac{1}{4}\right)^k \end{bmatrix}$$

$$= \begin{bmatrix} 48\left[1-\left(\frac{1}{2}\right)^n\right] - 108\left[1-\left(\frac{1}{3}\right)^n\right] + 64\left[1-\left(\frac{1}{4}\right)^n\right] \\ 24\left[1-\left(\frac{1}{2}\right)^n\right] - 36\left[1-\left(\frac{1}{3}\right)^n\right] + 16\left[1-\left(\frac{1}{4}\right)^n\right] \\ 12\left[1-\left(\frac{1}{2}\right)^n\right] - 12\left[1-\left(\frac{1}{3}\right)^n\right] + 4\left[1-\left(\frac{1}{4}\right)^n\right] \end{bmatrix}$$

$$y(n) = v_1(n) = 4 - 48\left(\frac{1}{2}\right)^n + 108\left(\frac{1}{3}\right)^n - 64\left(\frac{1}{4}\right)^n$$

$n \geq 0$

(ii) $$\underline{v}(n) = \sum_{k=0}^{n-1} \begin{bmatrix} \dfrac{\sin k\theta}{\sin\theta} \\ -\dfrac{\sin(k-1)\theta}{\sin\theta} - \dfrac{1}{\sqrt{2}} \dfrac{\sin k\theta}{\sin\theta} \end{bmatrix}$$

$$= \frac{1}{2j\sin\theta} \sum_{k=0}^{n-1} \begin{bmatrix} e^{jk\theta} - e^{-jk\theta} \\ -e^{j(k-1)\theta} + e^{-j(k-1)\theta} - \frac{1}{\sqrt{2}} e^{jk\theta} + \frac{1}{\sqrt{2}} e^{-jk\theta} \end{bmatrix}$$

$$= \frac{1}{2j\sin\theta} \begin{bmatrix} \dfrac{1-e^{jn\theta}}{1-e^{j\theta}} - \dfrac{1-e^{-jn\theta}}{1-e^{-j\theta}} \\ -e^{-j\theta}\dfrac{1-e^{jn\theta}}{1-e^{j\theta}} + e^{j\theta}\dfrac{1-e^{-jn\theta}}{1-e^{-j\theta}} - \dfrac{1}{\sqrt{2}}\dfrac{1-e^{jn\theta}}{1-e^{j\theta}} + \dfrac{1}{\sqrt{2}}\dfrac{1-e^{-jn\theta}}{1-e^{-j\theta}} \end{bmatrix}$$

$$= \frac{1}{2\sin\theta\,[1-\cos\theta]} \begin{bmatrix} \sin\theta - \sin n\theta + \sin(n-1)\theta \\ \left(1-\frac{1}{\sqrt{2}}\right)\sin\theta - \sin 2\theta + \left(1-\frac{1}{\sqrt{2}}\right)\sin(n-1)\theta + \frac{1}{\sqrt{2}}\sin n\theta \\ - \sin(n-2)\theta \end{bmatrix}$$

$$y(n) = -\frac{3}{4}v_1(n) + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)v_2(n) + x(n) \qquad n \geq 0$$

(iii) $$\underline{v}(n) = \sum_{k=0}^{n-1} \begin{bmatrix} 2^k - 1 \\ -1 + 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2^n - 1 - n \\ 2^{n+1} - 2 - n \end{bmatrix}$$

$$y(n) = 2^n - 1 - n \qquad n \geq 0$$

6.30 (a) $$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ 5/3 & \lambda + 7/6 \end{vmatrix} = \lambda^2 + \frac{1}{6}\lambda - \frac{1}{3} = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{2}{3}\right)$$

Characteristic roots are $\lambda = \frac{1}{2}, -\frac{2}{3}$

(b) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. Then $PA = \hat{A}P$ gives

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{5}{3} & -\frac{7}{6} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\left.\begin{aligned} p_{11} - \tfrac{5}{3}p_{12} &= \tfrac{1}{2}p_{11} \\ \tfrac{1}{2}p_{11} - \tfrac{7}{6}p_{12} &= \tfrac{1}{2}p_{12} \\ p_{21} - \tfrac{5}{3}p_{22} &= -\tfrac{2}{3}p_{21} \\ \tfrac{1}{2}p_{21} - \tfrac{7}{6}p_{22} &= -\tfrac{2}{3}p_{22} \end{aligned}\right\} \quad \begin{aligned} &\text{Gives } p_{11} = \tfrac{10}{3}p_{12} \\ &\qquad\quad p_{21} = p_{22} \\ &\text{Let } p_{11} = p_{22} = 1 \end{aligned}$$

$$\therefore\ P = \begin{bmatrix} 1 & 3/10 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 10/7 & -3/7 \\ -10/7 & 10/7 \end{bmatrix}$$

$$\hat{\underline{b}} = P^{-1}\underline{b} = \begin{bmatrix} 10/7 & -3/7 \\ -10/7 & 10/7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix}$$

$$\hat{c} = c\,P = [1 \quad 1]\begin{bmatrix} 1 & 3/10 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{13}{10} \end{bmatrix}$$

$$\hat{d} = d = 0$$

$$\hat{V}(0) = P^{-1}V(0) = \begin{bmatrix} \frac{10}{7}V_1(0) - \frac{3}{7}V_2(0) \\ -\frac{10}{7}V_1(0) + \frac{10}{7}V_2(0) \end{bmatrix}$$

(c)
$$\begin{bmatrix} \hat{V}_1(n+1) \\ \hat{V}_2(n+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} \hat{V}_1(n) \\ \hat{V}_2(n) \end{bmatrix} + \begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix} x(n)$$

$$\hat{A}^n = \begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & \left(-\frac{2}{3}\right)^n \end{bmatrix}. \qquad \text{Assume } \underline{\hat{V}}(0) = 0$$

$$\therefore \begin{bmatrix} \hat{V}_1(n) \\ \hat{V}_2(n) \end{bmatrix} = \sum_{j=0}^{n-1} \begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & \left(-\frac{2}{3}\right)^n \end{bmatrix}\begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{26}{7}\left[1 - \left(\frac{1}{2}\right)^n\right] \\ -\frac{12}{7}\left[1 - \left(-\frac{2}{3}\right)^n\right] \end{bmatrix} \qquad n \geq 0$$

(d)
$$\underline{V}(n) = P\,\underline{\hat{V}}(n) = \begin{bmatrix} \frac{16}{5} - \frac{26}{7}\left(\frac{1}{2}\right)^n + \frac{18}{35}\left(-\frac{2}{3}\right)^n \\ 2 - \frac{26}{7}\left(\frac{1}{2}\right)^n + \frac{12}{7}\left(-\frac{2}{3}\right)^n \end{bmatrix}$$

6.31 Consider the system described by

$$y_n + \sum_{j=1}^{N} a_j\, y(n-j) = \sum_{m=0}^{N} b_m\, x(n-m)$$

The characteristic equation for this system is

$$1 + \sum_{j=1}^{N} a_j z^{-j} = 0 \quad \text{or equivalently}$$

$$z^N + a_1 z^{N-1} + \dots + a_N = 0$$

The A matrix in the second form is given by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & \cdots & & 1 \\ -a_N & & \cdots & & -a_1 \end{bmatrix}$$

So that we must show that $\det|zI - A| = z^N + \sum_{j=1}^{N} a_j z^{N-j}$

We can prove this by induction. Clearly the result holds for $N = 1$ and 2 since $A = [a_1]$ and $A = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}$

Assume it holds for order $N-1$. Then for N, we have

$$\det[zI - A] = \begin{vmatrix} z & -1 & 0 & \cdots & & 0 \\ 0 & z & -1 & \cdots & & 0 \\ \vdots & & & & & \\ \cdot & & & & z & -1 \\ a_N & a_{N-1} & & \cdots & a_2 & z+a_1 \end{vmatrix}$$

$$= z \begin{vmatrix} z & -1 & \cdots & 0 \\ 0 & z & -1 & \cdots 0 \\ & & & \\ a_{N-1} & \cdots & & z+a_1 \end{vmatrix} + a_N$$

$$= z\left[z^{N-1} + \sum_{j=1}^{N-1} a_j z^{N-1-j}\right] + a_N$$

$$= z^N + \sum_{j=1}^{N} a_j z^{N-j}$$

6.32 (i) $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is not bounded

$\therefore$ unstable

(ii) $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{100} 3^n = \frac{1}{2}\left(3^{101} - 1\right) < \infty$

$\therefore$ stable

(iii) $\sum |h(n)| < \infty$ stable

(iv) $|y(n)| = |x(n) + 2x(n-1) + \frac{1}{2}x(n-2)|$

$\leq |x(n)| + 2|x(n-1)| + \frac{1}{2}|x(n-2)| < \infty$

if $|x(n)| < \infty$

Stable

(v) Ch. Eqn is $z^2 - 2z + 1 = 0$

Roots are $z = 1, 1$ unstable

(vi) $z^2 - \frac{2}{3}z - \frac{1}{3} = 0$, Roots are $z = -1, \frac{1}{3}$ Unstable

(vii) $\mathrm{Det}(zI - A) = \begin{vmatrix} z - \frac{1}{2} & -\frac{1}{4} \\ 1/4 & z - \frac{1}{2} \end{vmatrix} = z^2 - z + \frac{5}{16}$

Roots are $z = -\frac{1}{2} \pm \frac{1}{8}\sqrt{11}$ Stable

(viii) $\mathrm{Det}(zI - A) = \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix} = z^2 + 3z + 2$

Roots are $-2, -1$ Unstable

CHAPTER 7

7.1 (a) $\Omega_0 = \dfrac{3\pi}{4} \quad N = \dfrac{2\pi}{3\pi/4} = 8$

$$x(n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n}$$

$\therefore a_1 = -a_1 = a_2 = \frac{1}{2}$; All other $a_k = 0$
$0 \le k \le 7$

b) $x(n) = 2\left[\sin \dfrac{7\pi n}{6} - \sin \dfrac{\pi n}{6}\right]$

$N = 12, \quad \Omega_0 = \pi/6$

$\therefore x(n) = \dfrac{1}{j}\left[e^{j7\Omega_0 n} - e^{-j7\Omega_0 n} - e^{j\Omega_0 n} + e^{-j\Omega}\right.$

$\therefore a_1 = \dfrac{-1}{j} = j = a_{-1}^* = a_{11}^*$

$a_7 = -j = a_{-7}^* = a_5^*$

All others zero

c) $e^{j\frac{2\pi}{5}} = .3090 + j\,.9511$

$$a_k = \frac{1}{5}\left[1 - e^{j\frac{2\pi}{5}k} + e^{j\frac{6\pi}{5}k} - e^{+j\frac{8\pi}{5}k}\right]$$

$a_0 = 0 \quad a_1 = -.4271 - j\,0.5878 = a_4^*$

$a_2 = 2.9271 + j\,.9511 = a_3^*$

d) $e^{j\frac{2\pi}{8}} = \dfrac{1}{\sqrt{2}} + j\dfrac{1}{\sqrt{2}} = e^{j\pi/4}$

$$a_k = \frac{1}{8}\left[1 + e^{j\frac{\pi}{4}k} + e^{j\frac{\pi}{2}k} + e^{j\frac{3\pi}{4}k}\right]$$

$a_0 = \frac{1}{2}, \quad a_1 = .125 + j\,.7018 = a_7^*$

$a_2 = 0 \quad a_3 = .125 + j\,.0518 = a_5^*, \; a_4 = 0$

e) $e^{j\frac{2\pi}{6}} = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\pi/3}$

$a_k = \left[e^{j\pi/3 k} + 2e^{j\frac{2\pi}{3}k} + 3e^{j\pi k}\right]\frac{1}{8}$

$a_0 = 0.75 \quad a_1 = -3.5 + j\,2.5981$

$= a_5^* = -.4375 + j.3248$

$a_2 = .1875 - j.1083 = a_4^*, \; a_3 = -.25$

f) First term is periodic with period 2;
Second term is periodic with period 3.
$\therefore$ Period of the sum is $N = 6$, $\Omega_0 = \frac{2\pi}{6} = \pi/3$

Now $\cos\frac{2\pi n}{3} = \frac{1}{2}\left[e^{j2\Omega_0 n} + e^{-j2\Omega_0 n}\right]$

$\therefore a_2 = 1/2 = a_4$

Also the coefficients of $\sum_{k=-\infty}^{\infty}(-1)^k\delta(n-k)$

Considered as a periodic sequence with $N=6$

is $a_k = \frac{1}{6}\left[1 - e^{j\pi/3 k} + e^{j2\pi/3 k} - e^{+j\pi k} + e^{j4\frac{\pi}{3}k} - e^{j\frac{6\pi}{3}k}\right]$

so that $a_3 = 1$, all others zero

$\therefore$ For $x(n)$, we have $a_0 = a_1 = a_5 -$

$a_2 = a_4 = 1/2, \; a_3 = 1$

7.2(a) $a_k = 1 + 1/2\cos 2\Omega_0 k + 2\cos\Omega_0 k$

$= 1 + 1/4\,e^{-j2\Omega_0 k} + 1/4\,e^{-j6\Omega_0 k} + e^{-j\Omega_0 k} + e^{-j7\Omega_0 k}$

$\therefore x(0) = 8, \; x(1) = x(7) = 8, \; x(2) = x(6) = 2$

$x(3) = x(4) = x(5) = 0$

b) $x(n) = e^{j\frac{2\pi}{8}n} + e^{j\frac{2\pi}{8}\cdot 2n} + e^{j\frac{2\pi}{8}\cdot 3n}$

$= e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{2}n} + e^{j\frac{3}{4}\pi n}$

$\therefore\ x(0) = 3,\ x(1) = j2\cdot 4142,\ x(2) = -1$

$x(3) = j\cdot 4142,\ x(4) = -1,\ x(5) = -j0\cdot 4142$

$x(6) = -1,\ x(7) = -j2\cdot 4142$

c) $x(n) = \sum_{k=0}^{7} e^{-j\frac{\pi k}{4}} e^{j\frac{2\pi}{8}kn} = \sum_{k=0}^{7} e^{j\pi/4(n-1)k}$

$$= \begin{cases} \dfrac{1-e^{j2\pi(n-1)}}{1-e^{j\pi/4(n-1)}} & n \neq 1 \\ 0 & n = 1 \end{cases}$$

$$= \begin{cases} 0 & n \neq 1 \\ 0 & n = 1 \end{cases}$$

d) $x(n) = 1 - e^{-j4\pi/5\, n} + e^{-j8\pi/5\, n}$

$x(0) = 1,\ x(1) = 2\cdot 118 + j1\cdot 5388,$

$= x^*(4)$

$x(2) = -0\cdot 118 - j0\cdot 3633 = x^*(3)$

7.3 (i) $b_k = \frac{1}{N}\sum_{\langle N\rangle} x(n-n_0)e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{\langle N\rangle} x(m)\, e^{-jk\Omega_0(n+m)}$

$= a_k e^{-jk\Omega_0 n_0}$

(ii) $b_k = \frac{1}{N}\sum_{\langle N\rangle} x(-n)\, e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{\langle N\rangle} x(m)\, e^{-j(-k)\Omega_0 m} = a_{-k} = a_{N-k}$

(iii) Let N be even. Then $x(-n)$ is periodic with period N

$$b_k = \frac{1}{N}\sum_{<N>} (-1)^n x(n) e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{<N>} x(n)\, e^{j\pi n}\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{<N>} x(n) e^{-j(k-\frac{N}{2})\frac{2\pi}{N}n} = a_{k-\frac{N}{2}}$$

If N is odd, $x(-n)$ is periodic with period 2N.

$$b_k = \frac{1}{2N}\sum_{n=0}^{2N-1} (-1)^n x(n) e^{-jk\frac{\pi}{N}n}$$

$$= \frac{1}{2N}\left[\sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{k-N}{2}\right)\frac{2\pi}{N}n} + \sum_{n=N}^{2N-1} x(n) e^{-j\left(\frac{k-N}{2}\right)\frac{2\pi}{N}n}\right]$$

$$= \frac{1}{2}\left[a_{\left(\frac{k-N}{2}\right)} + \sum_{m=0}^{N-1} x(m+N) e^{-j\left(\frac{k-N}{2}\right)\frac{2\pi}{N}(m+N)}\right]$$

$$= \frac{1}{2}\left[a_{\left(\frac{k-N}{2}\right)} + e^{-j(k-N)\pi}\, a_{\left(\frac{k-N}{2}\right)}\right]$$

where the last step follows since $x(m)$ is periodic with period N. Since N is odd, $k-N$ is odd if k is even and $k-N$ is even if k is odd.

$$\therefore\ b_k = \begin{cases} a_{\left(\frac{k-N}{2}\right)} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

(iv) $y(n) = \frac{1}{2}x(n) + \frac{1}{2}(-1)^n x(n)$

$$\therefore\ b_k = \frac{1}{2}a_k + \frac{1}{2}a_{k-\frac{N}{2}} \quad \text{if } N \text{ is even}$$

$$b_k = \begin{cases} \frac{1}{2}a_k + \frac{1}{2}a_{\left(\frac{k-N}{2}\right)} & \text{for } k \text{ odd, } N \text{ odd} \\ 0 & \text{for } k \text{ even, } N \text{ odd} \end{cases}$$

(v) $y(n) = \frac{1}{2}x(n) - \frac{1}{2}(-1)^n x(n)$

$$b_k = \begin{cases} \frac{1}{2}a_k - \frac{1}{2}a_{k-\frac{N}{2}} & N \text{ even} \\ \frac{1}{2}a_k - \frac{1}{2}a_{\left(\frac{k-N}{2}\right)} & N \text{ odd, } k \text{ odd} \\ 0 & N \text{ odd, } k \text{ even} \end{cases}$$

(vi) $y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(-n)$

$$b_k = \frac{1}{2}a_k + \frac{1}{2}a_{N-k}$$

(vii) $y(n) = \frac{1}{2}x(n) - \frac{1}{2}x(-n)$

$$b_k = \frac{1}{2}a_k - \frac{1}{2}a_{N-k}$$

7.4 $$a_{N-k} = \frac{1}{N}\sum_{\langle N\rangle} x(n)e^{-j\Omega_0(N-k)n}$$

$$= \frac{1}{N}\sum_{\langle N\rangle} x(n)e^{j\Omega_0 kn} = \frac{1}{N}\left[\sum_{\langle N\rangle} x(n)e^{-j\Omega_0 kn}\right]^*$$

$$= a_k^*$$

7.5 $$H(\Omega) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\Omega n} = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

So that $$b_k = H(k\Omega_0)a_k = \frac{1}{1-\frac{1}{2}e^{-jk\Omega_0}}a_k$$

7.6 $$H(\Omega) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^{|n|} e^{-j\Omega n} = \frac{3/4}{1-\cos\Omega+\frac{1}{4}}$$

and $$b_k = \frac{3/4}{1-\cos k\Omega_0+\frac{1}{4}}a_k$$

7.7 (i) $$C_k = \frac{1}{N}\sum_n x(n)h(n)e^{-j\Omega_0 kn}$$

$$= \frac{1}{N}\sum_n x(n)\left[\sum_m b_m e^{j\Omega_0 mn}\right]e^{-j\Omega_0 kn}$$

$$= \sum_m\left[\frac{1}{N}\sum_n x(n)e^{-j\Omega_0(k-m)n}\right]b_m = \sum_{m=0}^{N-1} a_{k-m}b_m$$

$$= a_k \circledast b_k$$

(ii) $$y(n) = \sum_{m=0}^{N-1} x(m)h(n-m)$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m)h(n-m)\right]e^{-j\Omega_0 kn}$$

$$= \left[\frac{1}{N}\sum_{m=0}^{N-1} x(m)e^{-j\Omega_0 km}\right]\left[\sum_{n=0}^{N-1} h(n-m)e^{-j\Omega_0 k(n-m)}\right]$$

$$= N a_k b_k$$

7.8 (a) $a_k = \left[1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}\right]/6$

$b_k = \left[1 - e^{-j\pi/3 k} + e^{-j2\pi/3 k} - e^{-j4\pi/3 k} + e^{-j5\pi/3 k}\right]$

$a_0 = 1/2$, $a_1 = 0.1667$, $a_2 = j\,.2887$, $a_3 = .1667$
$a_4 = -j\,.2887$, $a_5 = .1667$

$b_0 = .1667$, $b_1 = 0.1667$, $b_2 = .1667 + .5774j$
$b_3 = .1667$, $b_4 = .1667 - j\,.5774$, $b_5 = .1667$
$a \quad a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

a_{11}
$c_0 = .5$, $c_1 = .1667$, $c_2 = j\,.2887 = c_4$
$c_3 = .1667 = c_5$

b) $x(n) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$ where $\Omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore a_0 = 0$, $a_1 = a_5 = \frac{1}{2}$, $a_2 = a_3 = a_4 = 0$

$b_k = \frac{1}{6}\left[1 - e^{-j\pi/3 k} + e^{-j2\pi/3 k} + e^{-j\pi k} - e^{-j4\pi/3 k} + e^{-j5\pi/3 k}\right]$

$b_0 = 1/3$, $b_1 = 0$, $b_2 = .3333 + j\,.5774 = b_4$, $b_3 = 0$

$c_0 = 0$, $c_1 = .1111 + j\,.0962$, $c_2 = 0$, $c_3 = -.1111$
$c_4 = 0$, $c_5 = .1111 - j\,0.0962$

c) $x(n) = e^{j\Omega_0 n} + e^{-j\Omega_0 n}$ where $\Omega_0 = \pi/2$

$\therefore a_0 = 0, \; a_1 = 1, \; a_2 = 0, \; a_3 = -1$

$$b_k = 1 - \frac{1}{2} e^{-j\pi/2 k} + \frac{1}{4} e^{-j\pi k} - \frac{1}{8} e^{-j 3\pi/2 k}$$

$b_0 = .1562, \; b_1 = .1875 + j\,0.0938 = b_3^*$

$b_2 = .4688$

$c_0 = .375 = c_2, \; c_1 = .625 = c_3$

d) $$a_k = \frac{1}{8} \sum_{n=0}^{7} e^{-j\, 2\pi/8\, nk} = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

$$b_k = \frac{1}{8}\left[1 + 2e^{-j\pi/4 k} + 3e^{-j\pi/2 k} + 4e^{-j 3\pi/4 k} + 4e^{-j\pi k} + 3e^{-j 5\pi/4 k} + 2e^{-j 3\pi/2 k} + e^{-j 7\pi/4 k}\right]$$

$b_0 = 2.5, \; b_1 = -.7286 - .3018j = b_7^*$

$b_2 = 0 = b_6$

$b_3 = -0.0214 - 0.0518j, \; b_4 = 0$

$= b_5^*$

$c_k = b_k$

7.9 $c_k = N a_k b_k$ where a_k and b_k are as in 7.8

(i) $c_k = \{0, 0, 3 - j3\sqrt{3}, 0, 3 + j3\sqrt{3}, 0\}$

(ii) $c_0 = \frac{119}{8}$, $c_1 = \frac{1}{8\sqrt{2}}\left[(1+\sqrt{2}) + j\,(1-\sqrt{2})\right]$

$c_2 = \frac{1}{8\sqrt{2}}\left[(-1+\sqrt{2}) + j\,(1-\sqrt{2})\right]$

$c_3 = \frac{1}{8}$, $c_4 = \frac{1}{8\sqrt{2}}\left[(-1+\sqrt{2}) - j\,(1-\sqrt{2})\right]$

$c_5 = \frac{1}{8\sqrt{2}}\left[(1+\sqrt{2}) - j\,(1-\sqrt{2})\right]$

(iii) $r = 0$, $\phi = 0, 1, \cdot, 5$

(iv) $c_k = \left[0, 0, \frac{15}{8}, 0\right]$

7.10 $$\frac{1}{2\pi}\int_0^{2} e^{j\Omega(n-k)}\,d\Omega = \begin{cases} 1 & n = k \\ \dfrac{e^{j2\pi(n-k)} - 1}{j2\pi(n-k)} & n \neq k \end{cases} = \delta(n-k)$$

7.11 $$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n)\,(-jn)\, e^{-j\Omega n}$$

$$\therefore \mathfrak{F}\{n\,x(n)\} = j\,\frac{dX(\Omega)}{d\Omega}$$

Repeat p times to get the desired result.

7.12 (i) $$X(\Omega) = \sum_{n=0}^{n_0} e^{-j\Omega n} = \frac{1 - e^{-j\Omega(n_0+1)}}{1 - e^{-j\Omega}}$$

$$= \frac{\sin \frac{\Omega(n_0+1)}{2}}{\sin \frac{\Omega}{2}}\, e^{-j\Omega \frac{n_0}{2}}$$

(ii) $\mathcal{F}\left\{\left(\frac{1}{3}\right)^{|n|}\right\} = \sum_{n=-\infty}^{-1}\left(\frac{1}{3}\right)^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n} e^{-j\Omega n}$

$$= \frac{\frac{1}{3}e^{j\Omega}}{1-\frac{1}{3}e^{j\Omega}} + \frac{1}{1-\frac{1}{3}e^{-j\Omega}} = \frac{4}{5-3\cos\Omega}$$

$$\mathcal{F}\left\{n\left(\frac{1}{3}\right)^{|n|}\right\} = j\frac{d}{d\Omega}\,\frac{4}{5-3\cos\Omega}$$

$$= j\frac{3\sin\Omega}{(5-3\cos\Omega)^2}$$

(iii) $\mathcal{F}\{a^n a(n)\} = \frac{1}{1-2e^{-j\Omega}}$

$$\therefore \mathcal{F}\left\{a^n \cos\Omega_0 n\, u(n)\right\} = \mathcal{F}\left\{\frac{1}{2}a^n e^{j\Omega_0 n}u(n) + \frac{1}{2}a^n e^{-j\Omega_0 n}u(n)\right\}$$

$$= \frac{1}{2}\,\frac{1}{1-\alpha e^{-j(\Omega-\Omega_0)}} + \frac{1}{2}\,\frac{1}{1-\alpha e^{-j(\Omega+\Omega_0)}}$$

$$= \frac{1-\alpha\cos\Omega_0\, e^{-j\Omega}}{1-2\alpha e^{-j\Omega}\cos\Omega_0 + \alpha^2 e^{-j2\Omega}}$$

(iv) $\mathcal{F}\{e^{j3n}\} = \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega-3-2\pi k)$

(v) $\mathcal{F}\{e^{j\frac{\pi}{8}n}\} = \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\Omega-k\frac{\pi}{8}\right)$

(vi) $x(n) = \frac{1}{4j}e^{j3n} - \frac{1}{4j}e^{-j3n} + 2e^{j\frac{\pi}{8}n} + 2e^{-j\frac{\pi}{8}n}$

$$X(\Omega) = \frac{1}{4j}\sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega-3-2\pi k) - \frac{1}{4j}\sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega+3-2\pi k)$$

$$+ 4\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\Omega-k\frac{\pi}{8}\right)$$

(vii) $X(\Omega) = \sum_{n=0}^{n_0-1} \alpha^n e^{-j\Omega n} = \dfrac{1-\left(\alpha e^{-j\Omega}\right)^{n_0}}{1-\alpha e^{-j\Omega}}$

(viii) $X(\Omega) = \mathrm{rect}\left(\dfrac{\Omega}{1/3}\right)$

(ix) $X(\Omega) = \dfrac{1}{2\pi}\,\mathrm{rect}\left(\dfrac{\Omega}{1/3}\right) * \mathrm{rect}\left(\dfrac{\Omega}{1/2}\right)$

$$= \begin{cases} \Omega + \frac{5}{12} & -\frac{5}{12} \le \Omega < -\frac{1}{12} \\ \frac{1}{3} & -\frac{1}{12} \le \Omega < \frac{1}{12} \\ -\Omega + \frac{5}{12} & \frac{1}{12} \le \Omega < \frac{5}{12} \\ 0 & \text{otherwise} \end{cases}$$

(x) $X(\Omega) = \dfrac{1}{2j}\,\mathrm{rect}\left(\dfrac{\Omega - \pi/2}{1/3}\right) - \dfrac{1}{2j}\,\mathrm{rect}\left(\dfrac{\Omega + \pi/2}{1/3}\right)$

(xi) $X(\Omega) = \dfrac{1}{1-ae^{-j\Omega}} + \dfrac{a e^{-j\Omega}}{(1-ae^{-j\Omega})^2} = \dfrac{1}{(1-ae^{-j\Omega})^2}$

7.13 (a) $x(n) = \dfrac{-j}{2}\, e^{j\pi/3} + \dfrac{1}{2}\, e^{j2\pi/3} + \dfrac{1}{2}\, e^{j4\pi/3} + \dfrac{j}{2}\, e^{j5\pi/3}$

$= \dfrac{1}{2}\cos\dfrac{2\pi n}{3} + \dfrac{1}{2}\sin\dfrac{\pi n}{3}$

b) $X(\Omega) = 4\dfrac{\left(e^{j5\Omega} - 4e^{-j5\Omega}\right)}{2j} + 2\dfrac{\left(e^{j3\Omega} + e^{-j3\Omega}\right)}{2}$

$= -j2\,\delta(n+5) + j2\,\delta(n-5) + \delta(n-6) + \delta(n+3)$

c) $X(\Omega) = \dfrac{1}{\left[1-\frac{1}{2}e^{-j\Omega}\right]^2} = 2e^{j\Omega} \times \dfrac{\frac{1}{2}e^{-j\Omega}}{\left[1-\frac{1}{2}e^{-j\Omega}\right]^2}$

$= 2(n+1)\left(\frac{1}{2}\right)^{n+1} u(n+1)$

d) $X(\Omega) = \dfrac{2}{1+\frac{1}{3}e^{-j\Omega}} + \dfrac{-1}{1-\frac{1}{4}e^{-j\Omega}}$

$x(n) = 2\left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$

7.14 $\displaystyle\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\Omega)e^{j\Omega n}\,d\Omega$

$\displaystyle = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\Omega)\left[\sum_{n=-\infty}^{\infty} x(n)\,e^{j\Omega n}\right]d\Omega$

$\displaystyle = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2\,d\Omega$

7.15 $H(\Omega) = \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$

(i) $X(\Omega) = \dfrac{1-\frac{1}{3}\cos\frac{\pi}{2}e^{-j\Omega}}{1-\frac{2}{3}\cos\frac{\pi}{2}e^{-j\Omega}+\frac{1}{9}e^{-j2\Omega}} = \dfrac{1}{1+\frac{1}{9}e^{-j2\Omega}}$

$\therefore\ Y(\Omega) = H(\Omega)\,X(\Omega) = \dfrac{1}{\left(1+\frac{1}{9}e^{-j2\Omega}\right)\left(1-\frac{1}{2}e^{-j\Omega}\right)}$

$= \dfrac{9/13}{1-\frac{1}{2}e^{-j\Omega}} + \dfrac{\frac{2}{13}e^{-j\Omega}+\frac{4}{13}}{1+\frac{1}{9}e^{-j2\Omega}}$

$$= \frac{9/13}{1-\frac{1}{2}e^{-j\Omega}} + \frac{4/13}{1+\frac{1}{9}e^{-j2\Omega}} + \frac{6}{13}\,\frac{1/3}{1+\frac{1}{9}e^{-j2\Omega}}$$

$$\therefore\ y(n) = \frac{9}{13}\left(\frac{1}{2}\right)^n + \frac{4}{13}\left(\frac{1}{3}\right)^n \cos\frac{\pi}{2}n + \frac{6}{13}\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{2}\right)^n \quad n \geq 0$$

where we have used

$$\mathfrak{F}\left\{a^n \cos\Omega_0 n\, u(n)\right\} = \frac{1 - a\cos\Omega_0 e^{-j\Omega}}{1 - 2a\cos\Omega_0 e^{-j\Omega} + a^2 e^{-j2\Omega}}$$

$$\mathfrak{F}\left\{a^n \sin\Omega_0 n\, u(n)\right\} = \frac{a\sin\Omega_0 e^{-j\Omega}}{1 - 2a\cos\Omega_0 e^{-j\Omega} + a^2 e^{-j2\Omega}}$$

(ii) $$X(\Omega) = \frac{\frac{1}{3}e^{-j\Omega}}{1+\frac{1}{9}e^{-j2\Omega}}$$

$$Y(\Omega) = \frac{\frac{1}{3}e^{-j\Omega}}{\left(1+\frac{1}{9}e^{-j2\Omega}\right)\left(1-\frac{1}{2}e^{-j\Omega}\right)}$$

$$= \frac{6/13}{1-\frac{1}{2}e^{-j\Omega}} + \frac{\frac{4}{39}e^{-j\Omega} - \frac{6}{13}}{1+\frac{1}{9}e^{-j2\Omega}}$$

$$\therefore\ y(n) = \frac{6}{13}\left(\frac{1}{2}\right)^n + \frac{12}{39}\left(\frac{1}{3}\right)^n \sin\frac{\pi}{2}n - \frac{6}{13}\left(\frac{1}{3}\right)^n \cos\frac{n\pi}{2}$$

for $n \geq 0$

(iii) $$X(\Omega) = \frac{8/9}{1-\frac{2}{3}\cos\Omega + \frac{1}{9}} = \frac{8/9}{\left(1-\frac{1}{3}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{j\Omega}\right)}$$

$$Y(\Omega) = \frac{8/9}{\left(1-\frac{1}{3}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{j\Omega}\right)} \cdot \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$= \frac{-2}{1-\frac{1}{3}e^{-j\Omega}} + \frac{\frac{2}{5}e^{j\Omega}}{1-\frac{1}{3}e^{j\Omega}} + \frac{16/5}{1-\frac{1}{2}e^{-j\Omega}}$$

$$y(n) = \begin{cases} \frac{6}{5}(3)^n & n<0 \\ -2\left(\frac{1}{3}\right)^n + \frac{16}{5}\left(\frac{1}{2}\right)^n & n \geq 0 \end{cases}$$

(iv) $$X(\Omega) = \frac{\frac{1}{3}e^{-j\Omega}}{(1-\frac{1}{3}e^{-j\Omega})^2}$$

$$Y(\Omega) = \frac{\frac{1}{3}e^{-j\Omega}}{(1-\frac{1}{3}e^{-j\Omega})^2(1-\frac{1}{2}e^{-j\Omega})}$$

$$= \frac{6}{1-\frac{1}{2}e^{-j\Omega}} + \frac{-2}{(1-\frac{1}{3}e^{-j\Omega})^2} - \frac{4}{1-\frac{1}{3}e^{-j\Omega}}$$

$$y(n) = 6\left(\tfrac{1}{2}\right)^n u(n) - 2(n+1)\left(\tfrac{1}{3}\right)^{n+1} u(n+1) - 4\left(\tfrac{1}{3}\right)^n u(n)$$

$$= 6\left(\tfrac{1}{2}\right)^n u(n) - \tfrac{2}{3}n\left(\tfrac{1}{3}\right)^n u(n) - \tfrac{14}{3}\left(\tfrac{1}{3}\right)^n u(n)$$

7.16 (a) $$X(\Omega) = \frac{1}{1+\frac{1}{9}e^{-j2\Omega}}$$

$$Y(\Omega) = \frac{e^{-j\Omega}}{(1+\frac{1}{9}e^{-j2\Omega})} + \frac{1}{(1+\frac{1}{9}e^{-j2\Omega})(1-\frac{1}{4}e^{-j\Omega})}$$

$$= \frac{e^{-j\Omega}}{(1+\frac{1}{9}e^{-j2\Omega})} + \frac{\frac{4}{25}e^{-j\Omega}+\frac{16}{25}}{1+\frac{1}{9}e^{-j2\Omega}} + \frac{9/25}{1-\frac{1}{4}e^{-j\Omega}}$$

$$= \frac{9/25}{1-\frac{1}{4}e^{-j\Omega}} + \frac{e^{-j\Omega}}{1+\frac{1}{9}e^{-j2\Omega}} + \frac{16/25}{1+\frac{1}{9}e^{-j2\Omega}} + \frac{12}{25}\,\frac{1/3}{1+\frac{1}{9}e^{-j2\Omega}}$$

$$y(n) = \frac{9}{25}\left(\frac{1}{4}\right)^n (n) + \left(\frac{1}{3}\right)^{n-1}\sin\frac{\pi n}{2}\,u(n-1) +$$

$$\left[\frac{16}{25}\left(\frac{1}{b}\right)^n\cos\frac{\pi}{2}n + \frac{12}{25}\left(\frac{1}{3}\right)^n\sin\frac{\pi}{2}n\right]u(n)$$

iii) $$X(\Omega) = \frac{8/9}{\left(1-\frac{1}{3}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{j\Omega}\right)}$$

$$Y(\Omega) = \frac{\frac{8}{9}e^{-j\Omega}}{1-\frac{2}{3}\cos\Omega+\frac{1}{9}} + \frac{8/9}{\left(1-\frac{1}{3}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}$$

$$= \frac{\frac{8}{9}e^{-j\Omega}}{1-\frac{2}{3}\cos\Omega+\frac{1}{9}} + \frac{4}{1-\frac{1}{3}e^{-j\Omega}} - \frac{\frac{12}{11}e^{j\Omega}}{1-\frac{1}{3}e^{j\Omega}}$$

$$- \frac{32/11}{1-\frac{1}{4}e^{-j\Omega}}$$

$$y(n) = \frac{8}{9}\left[\frac{1}{3}\left(\frac{1}{3}\right)^{-n}u(-n-1)+\frac{1}{3}\delta(n) + 3\left(\frac{1}{3}\right)^{n}u(n-1)\right]$$

$$+ 4\left(\frac{1}{3}\right)^{n}u(n) - \frac{12}{11}\left(\frac{1}{3}\right)^{-n}u(-n-1)$$

$$- \frac{32}{11}\left(\frac{1}{4}\right)^{n}u(n)$$

d) $$X(\Omega) = \frac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2}$$

$$Y(\Omega) = \frac{\frac{1}{3}e^{-j2\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2} + \frac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2\left(1-\frac{1}{4}e^{-j\Omega}\right)}$$

$$= \frac{\frac{1}{3}e^{-j2\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2} + \frac{4/3}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2} + \frac{-12}{1-\frac{1}{3}e^{-j\Omega}} + \frac{12}{1-\frac{1}{4}e^{-j\Omega}}$$

$$y(n) = (n-1)\left(\tfrac{1}{3}\right)^{n-1}u(n-1) + \tfrac{4}{3}(n+1)\left(\tfrac{1}{3}\right)^{n+1}u(n+1)$$

$$-12\left(\tfrac{1}{3}\right)^{n}u(n) + 12\left(\tfrac{1}{4}\right)^{n}u(n)$$

ii) $$X(\Omega) = \frac{\frac{1}{3} e^{-j\Omega}}{1 + \frac{1}{9} e^{-j2\Omega}}$$

$$Y(\Omega) = \frac{\frac{1}{3} e^{-j2\Omega}}{1 + \frac{1}{9} e^{-j2\Omega}} + \frac{\frac{1}{3} e^{-j\Omega}}{\left(1 + \frac{1}{9} e^{-j2\Omega}\right)\left(1 - \frac{1}{4} e^{-j\Omega}\right)}$$

$$= \frac{\frac{1}{3} e^{-j2\Omega}}{1 + \frac{1}{9} e^{-j2\Omega}} + \frac{-\frac{12}{25} + \frac{16}{75} e^{-j\Omega}}{1 + \frac{1}{9} e^{-j2\Omega}} + \frac{\frac{12}{25}}{1 - \frac{1}{4} e^{-j\Omega}}$$

$$y(n) = \left(\frac{1}{3}\right)^{n-1} \sin \frac{\pi(n-1)}{2}\, u(n-1) - \frac{12}{25}\left(\frac{1}{3}\right)^n \cos \frac{\pi n}{2}\, u(n)$$
$$+ \frac{48}{75}\left(\frac{1}{3}\right)^n \sin \frac{\pi n}{2}\, u(n) + \frac{12}{25}\left(\frac{1}{4}\right)^n u(n)$$

7.17 $H(\Omega) = \text{rect}\left(\frac{\Omega}{\pi/2}\right)$

(i) The Fourier Series Coefficients are

$$a_0 = \frac{2}{3}$$

$$a_k = \frac{1}{6} \sum_{n=0}^{3} e^{-j\frac{2\pi}{6}nk} = \frac{1}{6} \frac{1 - e^{-jk\frac{4\pi}{3}}}{1 - e^{-jk\frac{\pi}{3}}} \qquad k \neq 0$$

i.e $$a_k = \frac{1}{6} e^{-jk\pi/2} \frac{\sin 2\pi k/3}{\sin k\pi/6} \qquad \text{for } k \neq 0$$

Filter only passes frequencies in the range $|\Omega| < \frac{\pi}{4}$ so that only dc term is passed through.

Thus output spectrum is

$$Y(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \frac{2}{3} \delta(\Omega - 2\pi m)$$

and $y(n) = \frac{2}{3}$

(ii) $x(n)$ is periodic with $N=2$ and $x(0)=1$ $x(1)=-1$

$\therefore$ $a_0 = 0$

$\therefore$ $a_1 = 1$

$$\therefore \quad X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi m)$$

so that $Y(\Omega) = 0$ and $y(n) = 0$

7.18 Since $H(\Omega) = rect\left(\frac{\Omega}{3\pi/4}\right)$, in case (i),

Only dc & 1st harmonic are passed through

$$\therefore \quad Y(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \left[\frac{2}{3}\delta(\Omega - 2\pi m) + \frac{\sqrt{3}}{6j}\delta\left(\Omega - \frac{\pi}{3} - 2\pi m\right)\right.$$
$$\left. - \frac{\sqrt{3}}{6j}\delta\left(\Omega - \frac{5\pi}{3} - 2\pi m\right)\right]$$

and $y(n) = \frac{2}{3} + \frac{1}{\sqrt{3}}\sin\frac{\pi n}{3}$

For case (ii), again $Y(\Omega) = 0$ so that $y(n) = 0$

7.19 (i) $\left[1 + \frac{5}{8}e^{-j\Omega} + \frac{3}{22}e^{-j2\Omega}\right]Y(\Omega) = \left[1 - e^{-j\Omega}\right]X(\Omega)$

$$\therefore \quad H(\Omega) = \frac{1 - e^{-j\Omega}}{1 + \frac{5}{8}e^{-j\Omega} + \frac{3}{32}e^{-j2\Omega}}$$

$$= \frac{11}{1 + \frac{3}{8}e^{-j\Omega}} - \frac{10}{1 + \frac{1}{4}e^{-j\Omega}}$$

$$h(n) = 11\left(-\frac{3}{8}\right)^n u(n) - 10\left(-\frac{1}{4}\right)^n u(n)$$

(ii) $\left[1 + e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega}\right]\cancel{Y}(\Omega) = X(\Omega)$

$$H(\Omega) = \frac{1}{1 + e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}$$

$$= \frac{\frac{1-\sqrt{2}}{2}}{\left[1 + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)e^{-j\Omega}\right]} + \frac{\frac{1+\sqrt{2}}{2}}{\left[1 + \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)e^{-j\Omega}\right]}$$

$$h(n) = \left(\frac{1-\sqrt{2}}{2}\right)\left[-\frac{1}{2}+\frac{1}{2\sqrt{2}}\right]^n u(n) + \left(\frac{1+\sqrt{2}}{2}\right)\left(-\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)^n u(n)$$

(iv) $\left[1 - 3e^{-j\Omega} + 2e^{-j2\Omega}\right] Y(\Omega) = X(\Omega)$

$$H(\Omega) = \frac{1}{1-3e^{-j\Omega}+2e^{-j2\Omega}}$$

This is not a valid Fourier transform and so $H(\Omega)$ does not exist for this system

(v) $\left[e^{j2\Omega} - \frac{1}{12}e^{j\Omega} - \frac{1}{12}\right] Y(\Omega) = \left[1 + \frac{1}{2}e^{j\Omega}\right] X(\Omega)$

$$H(\Omega) = \frac{1+\frac{1}{2}e^{j\Omega}}{e^{j2\Omega}-\frac{1}{12}e^{j\Omega}-\frac{1}{12}}$$

$$= \frac{-3/2}{1+\frac{1}{4}e^{-j\Omega}} + \frac{2}{1-\frac{1}{2}e^{-j\Omega}}$$

$$y(n) = -\frac{3}{2}\left(-\frac{1}{4}\right)^n u(n) + 2\left(\frac{1}{2}\right)^n u(n)$$

7-20 $$H(\Omega) = \frac{\frac{1}{2}+\frac{1}{12}e^{-j\Omega}}{1+\frac{5}{6}e^{-j\Omega}+\frac{1}{6}e^{-j2\Omega}}$$

$$= \frac{-\frac{1}{2}}{1+\frac{1}{3}e^{-j\Omega}} + \frac{1}{1+\frac{1}{2}e^{-j\Omega}}$$

(a) $h(n) = -\frac{1}{2}\left(-\frac{1}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$

(b) Let $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$. Then

$\left[1+\frac{5}{6}e^{-j\Omega}+\frac{1}{6}e^{-j2\Omega}\right]Y(\Omega) = \left[\frac{1}{2}+\frac{1}{12}e^{-j\Omega}\right]X(\Omega)$

$\therefore\ y(n)+\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2) = \frac{1}{2}x(n)+\frac{1}{12}x(n-1)$

(c) $X(\Omega) = \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{\frac{1}{2} + \frac{1}{12}e^{-j\Omega}}{(1+\frac{1}{3}e^{-j\Omega})(1+\frac{1}{2}e^{-j\Omega})(1-\frac{1}{2}e^{-j\Omega})}$$

$$= \frac{-1/5}{1+\frac{1}{3}e^{-j\Omega}} + \frac{\frac{1}{2}}{1+\frac{1}{2}e^{-j\Omega}} + \frac{\frac{1}{5}}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\therefore y(n) = -\frac{1}{5}\left(-\frac{1}{3}\right)^n u(n) + \frac{1}{2}\left(-\frac{1}{2}\right)^n u(n) + \frac{1}{5}\left(\frac{1}{2}\right)^n u(n)$$

7.21 $$|H(\Omega)|^2 = \frac{\alpha + e^{-j\Omega}}{1+\beta e^{-j\Omega}} \cdot \frac{\alpha^* + e^{j\Omega}}{1+\beta^* e^{j\Omega}} = \frac{|\alpha|^2 + 1 + \alpha e^{j\Omega} + \alpha^* e^{-j\Omega}}{1+|\beta|^2 + \beta e^{-j\Omega} + \beta^* e^{j\Omega}}$$

clearly if we choose $\alpha^* = \beta$, $|H(\Omega)|^2 = 1$

More generally, if we choose $\alpha = \frac{1}{\beta}$, we get

$|H(\Omega)|^2 = \dfrac{1}{|\beta|^2}$ = Constant which is also an all pass function

7.22 (a)

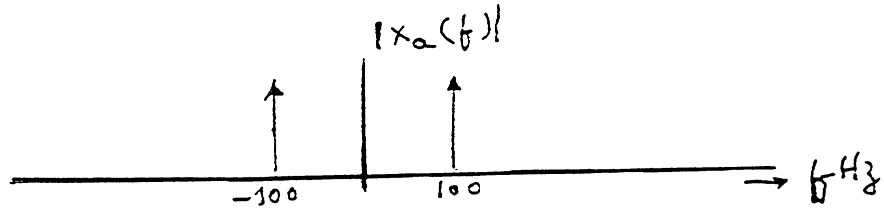
$$|H(\Omega)|^2 = \frac{1 + ae^{-j\Omega} + be^{-j2\Omega}}{b + ae^{-j\Omega} + e^{-j2\Omega}} \; \frac{1 + ae^{j\Omega} + be^{j2\Omega}}{b + ae^{j\Omega} + e^{j2\Omega}}$$

$= 1$

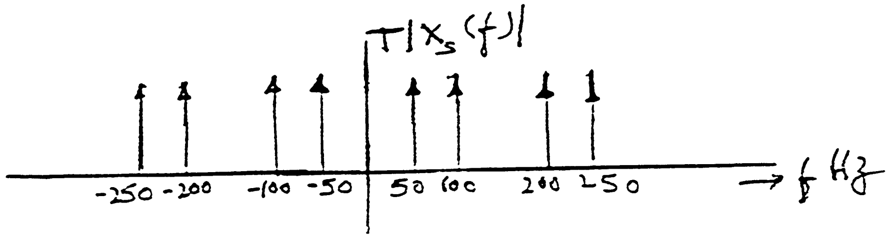
(b) Let $D(\Omega) = \sum_{k=0}^{N} a_k e^{-jk\Omega}$ with a_k real.

Then $|H(\Omega)|^2 = 1$ if $N(\Omega) = \sum_{k=0}^{N} a_{N-k} e^{-jk\Omega}$

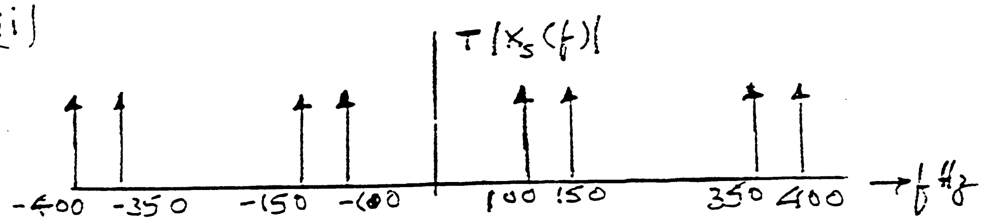
7.23



(a)(i)



(ii)



(b) In both cases, $x_a(t)$ can be recovered by using a band-pass filter centered around 100 Hz and with a bandwidth of say 50 Hz. In case (i), we can also use a low-pass filter with cutoff, say 125 Hz.

7.24 To find the impulse response, let $x_a(0) = 1$, with all other values zero. Then, for $-T \le t < 0$,

$$(t) = \hat{x}_0(t) = 1 + \frac{t}{T}$$

In the range $0 \le t < T$, $g_\ell(t) = -\frac{t+T}{T} = 1 - \frac{t}{T}$

$$\therefore g_\ell(t) = \begin{cases} 1 - \frac{|t|}{T} & |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{\sqrt{T}} \text{rect}\left(\frac{t}{T}\right) * \frac{1}{\sqrt{T}} \text{rect}\left(\frac{t}{T}\right)$$

$$\therefore G_\ell(\omega) = \frac{1}{T}\left[\frac{2 \sin \frac{\omega T}{2}}{\omega}\right]^2 = T\left[\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}}\right]^2$$

7.25 (a) $T = \frac{1}{10{,}000}$ s. Since the signal is bandlimited to 1.5 kHz, $f_s \geq 3000$ Hz for no aliasing.

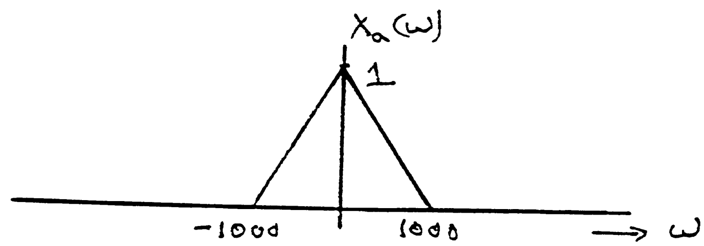
$\therefore \quad T' = MT < \frac{1}{3000}$

$M < \text{Int}\left[\frac{10{,}000}{3000}\right] = 3$

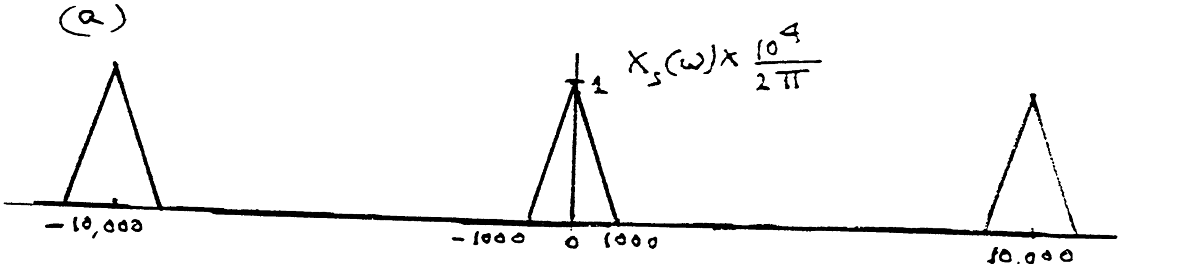
(b) $\frac{10{,}000}{4{,}000} = \frac{5}{2}$

$\therefore$ First interpolate by a factor of 5 and then decimate by a factor of 2.

7.26



(a)

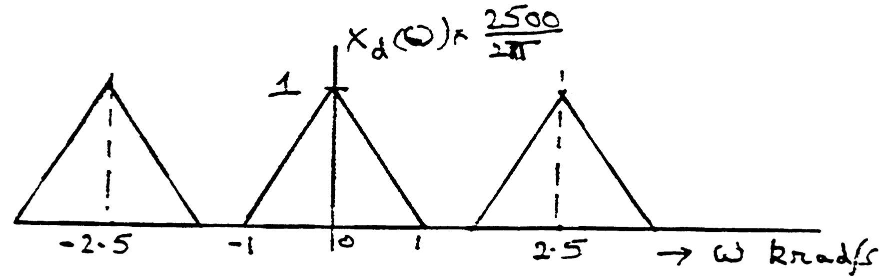


(b) Largest T' that can be used without aliasing distortion is $T' = \frac{2\pi}{2000} = MT = M \cdot \frac{2\pi}{10{,}000}$

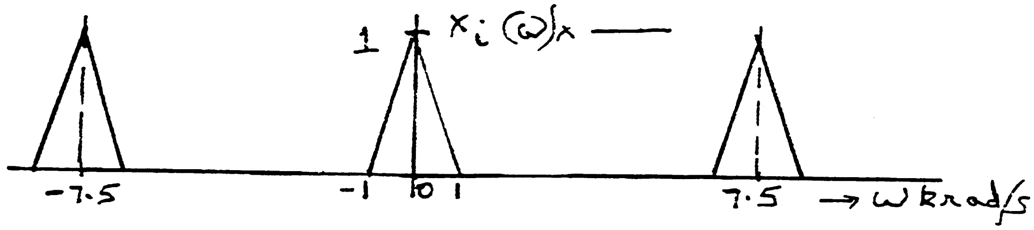
$\therefore \quad M = 5$

(c) $M = 4$ corresponds to $T' = 4 \times \frac{2\pi}{10{,}000}$ or a sampling frequency of $\omega_s = 2500$ rad/sec

(c)



(d)



7.27 From Eqn. (7.5.38), we have

$$\varepsilon(t) = \frac{\Delta}{T_2 - T_1}\left[t - \frac{T_2+T_1}{2}\right]$$

$$E = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \frac{\Delta^2}{(T_2-T_1)^2}\left[t - \frac{T_2+T_1}{2}\right]^2 dt$$

$$= \frac{\Delta^2}{(T_2-T_1)^3}\int_{-\frac{T_2-T_1}{2}}^{\frac{T_2-T_1}{2}} \tau^2\, d\tau = \frac{\Delta^2}{12}$$

7.28 (a) $(SNR)_{dB} = 10.79 + 10\log_{10} P_x + 6.02B - 20\log_{10} D$

$= 10.79 + 10\log_{10}(10) + 6.02B - 20\log_{10}(40)$

$= 5$

$\therefore B = 2.7.$ Use 3 bits.

(b) With $B = 3$ and $D = 2$, we have

$(SNR)_{dB} = 10.79 + 10\log_{10}(10) + 6.02\times 3 - 20\log_{10}(40)$

$= 12.83$ dB

7.29 (a) $10.79 + 10\log_{10}(10) + 6.02B - 20\log_{10}(40) = 10$ gives

$B = 3.53$. Set $B = 4$ bits

(b) With B as in part (a) and $D = 20$, we have

$(SNR)_{dB} = 10.79 + 10 + 6.02\times 4 - 20\log_{10}(20) = 18.85$ dB

Chapter 8

8.1 (a) $X(z) = \sum_{n=-\infty}^{-1} (-3)^n z^{-n} = \dfrac{z}{z+3}$; $|z| < 3$

(b) $X(z) = \sum_{n=-5}^{5} z^{-n}$ for $z \neq 0$

(c) $X(z) = \sum_{n=-\infty}^{-1} 3^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$

First term converges for $|z| < 3$, second for $|z| > \frac{1}{3}$.

$\therefore X(z) = \dfrac{-z}{z-3} + \dfrac{z}{z - \frac{1}{3}}$; $\frac{1}{3} < |z| < 3$

(d) $X(z) = 2 - \sum_{n=0}^{\infty} 2^n z^{-n} = 2 - \dfrac{z}{z-2} = \dfrac{z-4}{z-2}$; $|z| > 2$

8-2 $X(z) = \dfrac{z^3 + 4z^2 - \frac{11}{6}}{z^3 + \frac{7}{6}z^2 - \frac{3}{2}z + \frac{1}{3}}$

$= \dfrac{(z + 3.8781)(z + 0.7512)(z - 0.6292)}{(z - \frac{1}{3})(z - \frac{1}{2})(z+2)}$

$= \dfrac{1/3}{z - \frac{1}{3}} + \dfrac{\frac{1}{2}}{z - 1/2} - \dfrac{2}{z+2} + 1$

$= \dfrac{z}{z - 1/3} + \dfrac{z}{z - 1/2} - \dfrac{z}{z+2}$

(i) If ROC is $|z| < \frac{1}{3}$, all the poles are non causal.

$$\therefore Y(z) = -\left(\tfrac{1}{3}\right)^n u(-n) - \left(\tfrac{1}{2}\right)^n u(-n) + (-2)^n u(-n)$$

(ii) If ROC is $|z| > 3$, all the poles are causal

$$\therefore Y(z) = \left(\tfrac{1}{3}\right)^n u(n) + \left(\tfrac{1}{2}\right)^n u(n) - (-2)^n u(n)$$

8.3 (a)

$$Z[a^n \sin\Omega_o n] = \frac{az \sin\Omega_o}{z^2 - 2az\cos\Omega_o + a^2}$$

$$\therefore Z[n a^n \sin\Omega_o n] = -z \frac{d}{dz}\left[\frac{az\sin\Omega_o}{z^2 - 2(a\cos\Omega_o)z + a^2}\right]$$

$$= \frac{a\sin\Omega_o z^3 - a^3 \sin\Omega_o z}{[z^2 - 2az\cos\Omega_o + a^2]^2}$$

(b)

$$Z[n\cos\Omega_o n] = -z\frac{d}{dz}\,\frac{z(z-\cos\Omega_o)}{z^2 - 2z\cos\Omega_o + 1}$$

$$= \frac{z^3\cos\Omega_o - 2z^2 + z\cos\Omega_o}{[z^2 - 2z\cos\Omega_o + 1]^2}$$

(c)

$$X(z) = \frac{\frac{1}{2}z}{(z-1)^2} + \frac{1}{3}z^{-1}\,\frac{\frac{1}{3}z}{(z-1/3)^2}$$

$$= \frac{\frac{1}{2}z}{(z-1)^2} + \frac{1/9}{(z-1/3)^2}$$

(d)

$$X(z) = z^{-2} + \frac{z}{(z-1)^2}$$

(e)

$$Z[2\sin\tfrac{2}{5}\pi n] = \frac{2z\sin\frac{2}{5}\pi}{z^2 - 2z\cos\frac{2}{5}\pi + 1}$$

$$Z\left[2e^{-n}\sin\frac{2}{5}\pi n\right] = \frac{2(e^{-1}z)\sin\frac{2}{5}\pi}{(e^{-1}z)^2 - 2(e^{-1}z)\cos\frac{2\pi}{5} + 1}$$

8.4 (a) $x(nT) = nT\cos 1000\pi nT$

$$X(z) = \frac{T\left[z^3\cos(1000\pi T) - 2z^2 + z\cos(1000\pi T)\right]}{\left[z^2 - 2z\cos(1000\pi T) + 1\right]^2}$$

from Prob 8.3 (c) with $\Omega_0 = 1000\pi T$

(b) $x(nT) = nT\exp[-3(nT-1)] = Te^3 n e^{-3Tn}$

$$X(z) = \frac{e^3 T z\exp[-3T]}{\left[z - \exp[-3T]\right]^2}$$ from entry 16 in Table 8-2

8.5 (i)

$$\begin{array}{r|l} & 1 + \frac{5}{4}z^{-1} + \frac{13}{16}z^{-2} + \cdots \\ \hline 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} & 1 + \frac{1}{2}z^{-1} \\ & 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \\ \hline & \frac{5}{4}z^{-1} - \frac{1}{8}z^{-2} \\ & \frac{5}{4}z^{-1} - \frac{15}{16}z^{-2} + \frac{5}{32}z^{-3} \\ \hline & \frac{13}{16}z^{-2} - \frac{5}{32}z^{-3} \end{array}$$

etc.

$\therefore\ x(0) = 1,\quad x(1) = \frac{5}{4},\quad x(2) = \frac{13}{16}$ etc.

By Partial Fractions:

$$X(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} + \frac{-3}{1 - \frac{1}{4}z^{-1}}$$

$$\therefore\ x(n) = 4\left(\tfrac{1}{2}\right)^n u(n) - 3\left(\tfrac{1}{4}\right)^n u(n)$$

(ii) $$X(z) = \frac{z^2 + \frac{3}{4}z + \frac{1}{8}}{z^2 - \frac{5}{8}z + \frac{3}{32}}$$

By long division:

$$
\begin{array}{r|l}
 & 1 + \frac{11}{8} z^{-1} + \frac{57}{64} z^{-2} + \cdots \\
\hline
z^2 - \frac{5}{8} z + \frac{3}{32} & z^2 + \frac{3}{4} z + \frac{1}{8} \\
 & z^2 - \frac{5}{8} z + \frac{3}{32} \\
\cline{2-2}
 & \frac{11}{8} z + \frac{1}{32} \\
 & \frac{11}{8} z - \frac{55}{64} + \frac{33}{256} z^{-1} \\
\cline{2-2}
 & \frac{57}{64} - \frac{33}{256} z^{-1}
\end{array}
$$

$\therefore\ x(0) = 1, \quad x(1) = \frac{11}{8}, \quad x(2) = \frac{57}{64} \quad \cdots$

By Partial Fractions:

$$X(z) = \frac{z\left(z+\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}{z\left(z-\frac{3}{8}\right)\left(z-\frac{1}{4}\right)}$$

$$= \frac{4}{3} + \frac{35/3\ z}{z - 3/8} + \frac{-12 z}{z - 1/4}$$

$$x(n) = \frac{4}{3}\,\delta(n) + \left[\frac{35}{3}\left(\frac{3}{8}\right)^n - 12\left(\frac{1}{4}\right)^n\right] u(n)$$

(iii)

$$
\begin{array}{r|l}
 & z^{-3} + \frac{3}{2} z^{-4} + \frac{3}{2} z^{-5} + \frac{5}{4} z^{-6} + \cdots \\
\hline
z^3 - \frac{3}{2} z^2 + \frac{3}{4} z - \frac{1}{8} & 1 \\
 & 1 - \frac{3}{2} z^{-1} + \frac{3}{4} z^{-2} - \frac{1}{8} z^{-3} \\
\cline{2-2}
 & \frac{3}{2} z^{-1} - \frac{3}{4} z^{-2} + \frac{1}{8} z^{-3} \\
 & \frac{3}{2} z^{-1} - \frac{9}{4} z^{-2} + \frac{9}{8} z^{-3} - \frac{3}{16} z^{-4} \\
\cline{2-2}
 & \frac{3}{2} z^{-2} - z^{-3} + \frac{3}{16} z^{-4} \\
 & \frac{3}{2} z^{-2} - \frac{9}{4} z^{-3} + \frac{9}{8} z^{-4} - \frac{3}{16} z^{-5} \\
\cline{2-2}
 & \frac{5}{4} z^{-3} - \frac{15}{16} z^{-4} + \frac{3}{16} z^{-5}
\end{array}
$$

$x(0) = 0, \quad x(1) = 0, \quad x(2) = 0, \quad x(3) = 1, \quad x(4) = 3/2$

$x(5) = 3/2, \quad x(6) = 5/4$ etc.

Entry 9 in Table 8.4.2 gives, with $m = 2$,

$$x(n) = \frac{(n-2)(n-1)}{2!} \left(\tfrac{1}{2}\right)^{n-3} u(n-3)$$

(iv)

$$
\begin{array}{r l}
 & 1 - 2z^{-1} + 5z^{-2} - 14z^{-3} + \cdots \\
z^2 + 4z + 3 & \overline{\big)\, z^2 + 2z} \\
 & z^2 + 4z + 3 \\
\hline
 & -2z - 3 \\
 & -2z - 8 - 6z^{-1} \\
\hline
 & 5 + 6z^{-1} \\
 & 5 + 20z^{-1} + 15z^{-2} \\
\hline
 & -14z^{-1} - 15z^{-2}
\end{array}
$$

$\therefore \; x(0) = 1, \quad x(1) = -2, \quad x(2) = 5, \quad x(3) = -14$ etc.

By partial fractions:

$$X(z) = \frac{z(z+2)}{(z+1)(z+3)} = \frac{\frac{1}{2}z}{z+1} + \frac{\frac{1}{2}z}{z+3}$$

$$x(n) = (-1)^n \tfrac{1}{2} u(n) + \tfrac{1}{2}(-3)^n u(n)$$

8.6 (i) $\displaystyle \log\left(1 - \tfrac{1}{3} z^{-1}\right) = -\sum_{i=1}^{\infty} \frac{\left(\frac{1}{3} z^{-1}\right)^i}{i}$

$$= -\left[\tfrac{1}{3} z^{-1} + \tfrac{1}{18} z^{-2} + \cdots \right]$$

$\therefore \; x(0) = 0, \quad x(1) = -\frac{1}{3}$ etc.

In general $x(n) = -\frac{1}{n}\left(\frac{1}{3}\right)^n$ for $n \neq 0$

(ii) Let $y(n) = n x(n)$

Then $Y(z) = -z \frac{d}{dz} X(z) = -z \frac{1}{1-\frac{1}{3}z^{-1}} (-\frac{1}{3})(-z^{-2})$

$= -\frac{1}{3} \frac{1}{z - 1/3}$

$\therefore y(n) = -\frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u(n-1)$

$= -\left(\frac{1}{3}\right)^n u(n-1)$

$\therefore x(n) = \frac{1}{n} y(n) = -\frac{1}{n} \left(\frac{1}{3}\right)^n u(n-1)$

8.7 (i) $H(z) = \frac{z}{z-1/2}$, $X(z) = \frac{z(1-z^{-11})}{z-1}$

$\therefore Y(z) = \frac{z^2(1-z^{-11})}{(z-1/2)(z-1)} = (1-z^{-11})\left[\frac{-z}{z-1/2} + \frac{2z}{z-1}\right]$

$y(n) = \left(-\frac{1}{2}\right)^n [u(n) - u(n-11)] + 2[u(n) - u(n-11)]$

(ii) $H(z) = \frac{z}{z-1/3}$, $X(z) = \frac{z[1-z^{-10}]}{z-1} + \frac{z[1-z^{-6}]}{z-1}$

$\therefore Y(z) = \frac{z^2}{(z-1)(z-\frac{1}{3})} \left[(1-z^{-10}) + (1-z^{-6})\right]$

$= \left[\frac{3/2\, z}{z-1} - \frac{1/2\, z}{z-1/3}\right]\left[(1-z^{-10}) + (1-z^{-6})\right]$

$= \left[\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right]\left[2u(n) - u(n-6) - u(n-10)\right]$

(iii) $H(z) = [1 - z^{-1} + 2z^{-2} - z^{-3} + z^{-4}]$

$X(z) = [1 - 2z^{-2} + 3z^{-3}]$

$Y(z) = H(z)X(z) = 1 - z^{-1} + 4z^{-3} - 6z^{-4} + 8z^{-5} - 5z^{-6} + 3z^{-7}$

$\therefore y(n) = \{1, -1, 0, 4, -6, 8, -5, 3\}$

8.8 $H(z) = \dfrac{z - 1/2}{(z + 1/2)(z + 1/4)}$ $\qquad X(z) = \dfrac{z}{z-1}$

$$Y(z) = \frac{z(z - 1/2)}{(z-1)(z+\frac{1}{2})(z+1/4)}$$

$$= \frac{\frac{4}{15} z}{z-1} - \frac{8/3\, z}{z+1/2} + \frac{\frac{12}{5} z}{z + 1/4}$$

$$y(n) = \frac{4}{15} u(n) - \frac{8}{3}\left(-\frac{1}{2}\right)^n u(n) + \frac{12}{5}\left(-\frac{1}{4}\right)^n u(n)$$

8.9 (i) Transforming the difference equation gives

$$Y(z) - z^{-1}[Y(z) + z] + z^{-2}[Y(z) + z] = \frac{z}{z - \frac{1}{2}}$$

$$[1 - z^{-1} + z^{-2}]Y(z) = (1 - z^{-1}) + \frac{z}{z-\frac{1}{2}}$$

$$[z^2 - z + 1]Y(z) = z^2 - z + \frac{z^3}{z - \frac{1}{2}}$$

$$Y(z) = \frac{z(z-1)}{z^2 - z + 1} + \frac{z^3}{(z^2 - z + 1)(z - \frac{1}{2})}$$

$$= \frac{z(z-1)}{z^2 - z + 1} + \frac{\frac{2}{3} z(z+1)}{z^2 - z + 1} + \frac{\frac{1}{3} z}{z - 1/2}$$

$$= \frac{\frac{5}{3} z^2 - \frac{1}{3} z}{z^2 - z + 1} + \frac{\frac{1}{3} z}{z - 1/2}$$

$$= \frac{\frac{5}{3} z(z - \frac{1}{2})}{z^2 - z + 1} + \frac{1}{\sqrt{3}} \frac{\sqrt{3}/2\, z}{z^2 - z + 1} + \frac{1}{3}\frac{z}{z - 1/2}$$

$$y(n) = \left[\frac{5}{3}\cos\frac{\pi}{3}n + \frac{1}{\sqrt{3}}\sin\frac{\pi}{3}n + \frac{1}{3}\left(\frac{1}{2}\right)^n\right]u(n)$$

(b) Transform the difference equation to get

$$Y(z) - \frac{1}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) = z/z-\tfrac{1}{3}$$

$$\therefore Y(z) = \frac{z^2}{z^2 - \frac{1}{6} z - \frac{1}{6}} \; \frac{z}{z - \frac{1}{3}} = \frac{\frac{9}{5} z}{z - \frac{1}{2}} + \frac{\frac{1}{5} z}{z + \frac{1}{3}} - \frac{z}{z - \frac{1}{3}}$$

$$y(n) = \left[\frac{9}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}\left(-\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^n\right] u(n)$$

8.10 (i) $\left[1 + z^{-1} + z^{-2}\right] Y(z) - 1 = \dfrac{1}{1 - z^{-1}}$

$$Y(z) = \frac{z^2}{z^2 + z + 1} + \frac{z^3}{(z-1)(z^2 + z + 1)}$$

$$= \frac{\frac{5}{3} z^2 + \frac{1}{3} z}{z^2 + z + 1} + \frac{1}{3} \frac{z}{z - 1}$$

$$= \frac{5}{3} \frac{z^2 + \frac{1}{2} z}{z^2 + z + 1} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2} z}{z^2 + z + 1} + \frac{1}{3} \frac{z}{z - 1}$$

$$\therefore y(n) = \frac{5}{3} \cos \frac{2\pi}{3} n - \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3} n + \frac{1}{3} u(n) \qquad n \geq 0$$

(ii) $\left[1 - \frac{1}{4} z - \frac{1}{8} z^2\right] Y(z) - \frac{1}{4} - \frac{1}{8} z^{-1}$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{\frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$\therefore Y(z) = \frac{\frac{1}{4} z^2 + \frac{1}{8} z}{z^2 - \frac{1}{4} z - \frac{1}{8}} + \frac{z^2}{z^2 - \frac{1}{4} z - \frac{1}{8}}$$

$$= \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{4} z}{z + \frac{1}{4}}$$

$$y(n) = \left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n \qquad n \geq 0$$

(iii) $\left[1 - z^{-1} + \frac{15}{64} z^{-2}\right] Y(z) + \frac{15}{64} = \dfrac{1}{1 - 2z^{-1}}$

$$Y(z) = -\frac{\frac{15}{64} z^2}{z^2 - z + \frac{15}{64}} + \frac{z^2}{(z-2)(z^2 - z + \frac{15}{64})}$$

$$= \frac{\frac{256}{143} z}{z - 2} + \frac{\frac{12771}{18304} z}{z - \frac{3}{8}} + \frac{\frac{31525}{18304} z}{z - \frac{5}{8}}$$

$$y(n) = \frac{256}{143}(2)^n + \frac{12771}{18304}\left(\frac{3}{8}\right)^n + \frac{31525}{18304}\left(\frac{5}{8}\right)^n$$

(iv) $\left[z^2 + \frac{2}{3}z + \frac{1}{9}\right]Y(z) - z = \frac{z}{z - 1/2}$

$$Y(z) = \frac{z^2 + \frac{1}{2}z}{z^2 + \frac{2}{3}z + \frac{1}{9}}$$

$$= \frac{36/25\, z}{z - 1/2} + \frac{-36/25\, z}{(z - 1/3)} + \frac{-1/5\, z}{(z - 1/3)^2}$$

$$y(n) = \frac{36}{25}\left(\frac{1}{2}\right)^n - \frac{36}{25}\left(\frac{1}{3}\right)^n - \frac{3}{5}n\left(\frac{1}{3}\right)^n \qquad n \geq 0$$

(v) $Y(z) = \left[1 - 3z^{-1} + 2z^{-2} - z^{-3}\right]\frac{z}{z-1}$

$$y(n) = u(n) - 3u(n-1) + 2u(n-2) - u(n-3)$$

$$= \begin{cases} 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ -1 & n \geq 3 \end{cases}$$

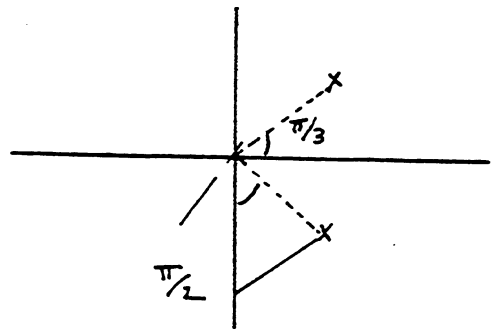
8.11 (i) $H(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} = \dfrac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$

$$= \frac{z(z - \frac{1}{4})}{z^2 - \frac{1}{2}z + \frac{1}{4}} + \frac{1/4\, z}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

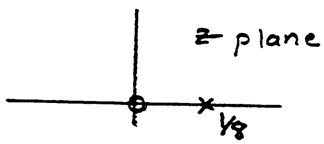
Use entries 10 and 11 in Table 8.4.2 with

$a = \frac{1}{2}$, $\Omega_0 = \frac{\pi}{3}$

$$\therefore h(n) = \left[\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{3} + \frac{1}{4}\left(\frac{1}{2}\right)^n \sin\frac{n\pi}{3}\right]u(n)$$



(ii) $$H(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{3}{8}z^{-1} + \frac{1}{32}z^{-2}} = \frac{z}{z - \frac{1}{8}}$$



$$h(n) = \left(\frac{1}{8}\right)^n u(n)$$

8.12 (a) $$X(z) = \frac{z}{z-1} + \frac{z}{z+\frac{1}{2}} = \frac{2z(z-\frac{1}{4})}{(z-1)(z+\frac{1}{2})}$$

$$Y(z) = \frac{6z}{z+\frac{1}{4}} - \frac{6z}{z+\frac{1}{3}} = \frac{\frac{1}{2}z}{(z+\frac{1}{4})(z+\frac{1}{3})}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}\,\frac{(z-1)(z+\frac{1}{2})}{(z+\frac{1}{4})(z-\frac{1}{4})(z+\frac{1}{3})}$$

$$= \frac{\frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} - \frac{1}{8}z^{-3}}{1 + \frac{1}{3}z^{-1} - \frac{1}{16}z^{-2} - \frac{1}{48}z^{-3}}$$

(b) The difference equation for the system is

$$y(n) + \frac{1}{3}y(n-1) - \frac{1}{16}y(n-2) - \frac{1}{48}y(n-3)$$
$$= \frac{1}{4}x(n-1) - \frac{1}{8}x(n-2) - \frac{1}{8}x(n-3)$$

8.13 $H_1(z) = \dfrac{z}{(z-1)^2} - \dfrac{z}{z-1} = \dfrac{2z - z^2}{(z-1)^2}$

$H_2(z) = 1 + \dfrac{z}{(z-1)^2} + z^{-2} = \dfrac{z^4 - z^3 + 2z^2 - 2z + 1}{z^2(z-1)^2}$

$H_3(z) = \dfrac{z}{z - 1/2}$

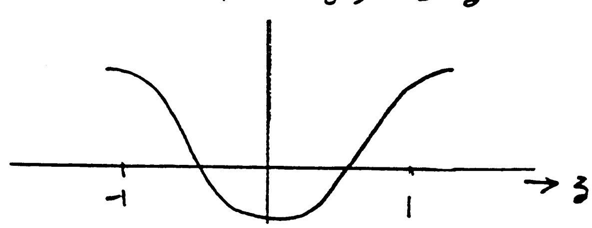
$H(z) = H_3(z)\left[H_2(z) - H_1(z) \right]$

$= \dfrac{z}{z - 1/2}\left[\dfrac{2z^4 - 3z^3 + 2z^2 - 2z + 1}{z^2(z-1)^2} \right]$

$= \dfrac{2z^3 - z^2 + z - 1}{z(z - 1/2)(z-1)}$

8.14 (a) Assuming a_1 and a_2 are real, $F(z)$ will have (i) both roots real or (ii) roots form a complex conjugate pair.

In the first case, for roots within the unit circle, the sketch of $F(z)$ vs z will be



$\therefore F(-1) > 0, \quad F(1) > 0$

Now with both roots complex conjugate,

$F(z) = (z - re^{j\theta})(z - re^{-j\theta})$

$= z^2 - 2rz\cos\theta + r^2$

For roots inside unit circle $|r| < 1$

$\therefore |F(0)| = r^2 < 1$

(b) $F(z) = z^2 + (.8K - 1.3)z + .04$

For stability

$F(-1) = 1 - (.8K - 1.3) + .04 = 2.34 - .8K > 0$

or $K < 2.925$

$F(1) = 1 + (.8K - 1.3) + .04 > 0$

or $K > .325$

Since $|F(0)| < 1$ for all K, system is stable if $.325 < K < 2.925$

8.15 Let $F(z) = z^2 + (K - 1 - \alpha)z + (1-K)\alpha$

$|F(0)| = |1 + K - 1 - \alpha + (1-K)\alpha| = |K(1-\alpha)|$

$F(-1) = 1 - (K - 1 - \alpha) + (1-K)\alpha = 2(1+\alpha) - K(1+\alpha)$

$F(1) = 1 + (K - 1 - \alpha) + (1-K)\alpha = K(1-\alpha)$

$|F(0)| < 1$ gives $-1 < K(1-\alpha) < 1$

$F(1) > 0$ gives $K(1-\alpha) > 0$

$\therefore \quad 0 < K(1-\alpha) < 1 \quad$ (i)

$F(-1) > 0$ gives $2(1+\alpha) > K(1+\alpha)$

If $\alpha > -1$, we get $K < 2$ (ii)

If $\alpha < -1$, we get $K > 2$ (iii)

If $K > 2$, from (i), $1 - \alpha < \frac{1}{K} < \frac{1}{2}$

$\therefore \alpha > \frac{1}{2}$ which contradicts (iii)

If $0 < K < 2$, from (i) we have $1 - \alpha > 0$

$\therefore \alpha < 1$

Also $K < \frac{1}{1-\alpha} \quad \therefore \quad 2 < \frac{1}{1-\alpha}$ or $1 - \alpha < \frac{1}{2}$

$\therefore \alpha > \frac{1}{2}$

$\therefore$ For $0 < K < 2$, $\frac{1}{2} < \alpha < 1$

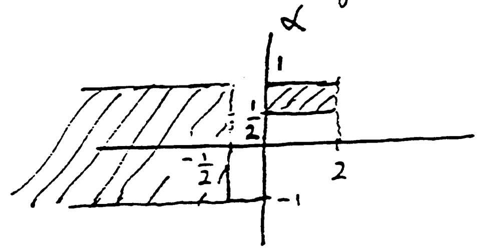
For $K < 0$, from (i) $1 - \alpha > 0$ $\therefore \alpha < 1$

From (ii), $\alpha > -1$. $\therefore -1 < \alpha < 1$

Also from (i) $1 - \alpha > \frac{1}{K}$ $\therefore -\infty < K < -\frac{1}{2}$

$\therefore$ For stability, if $0 < K < 2$, $\frac{1}{2} < \alpha < 1$

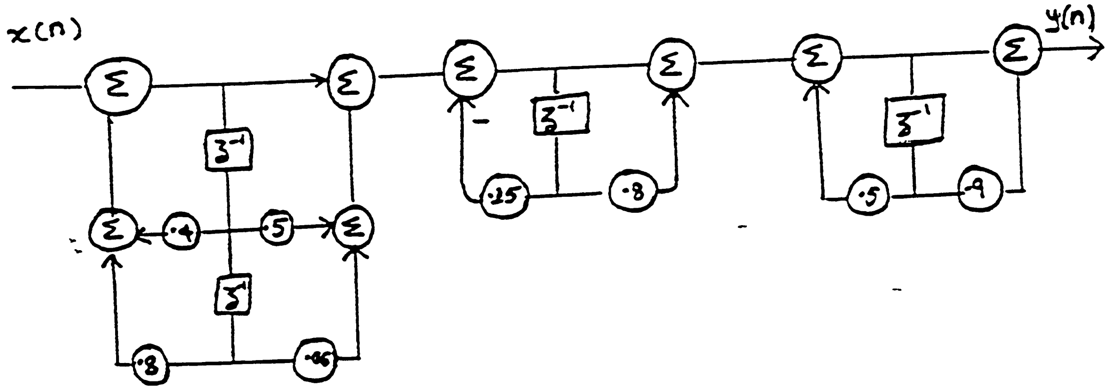
if $-\infty < K < -\frac{1}{2}$, $-1 < \alpha < 1$



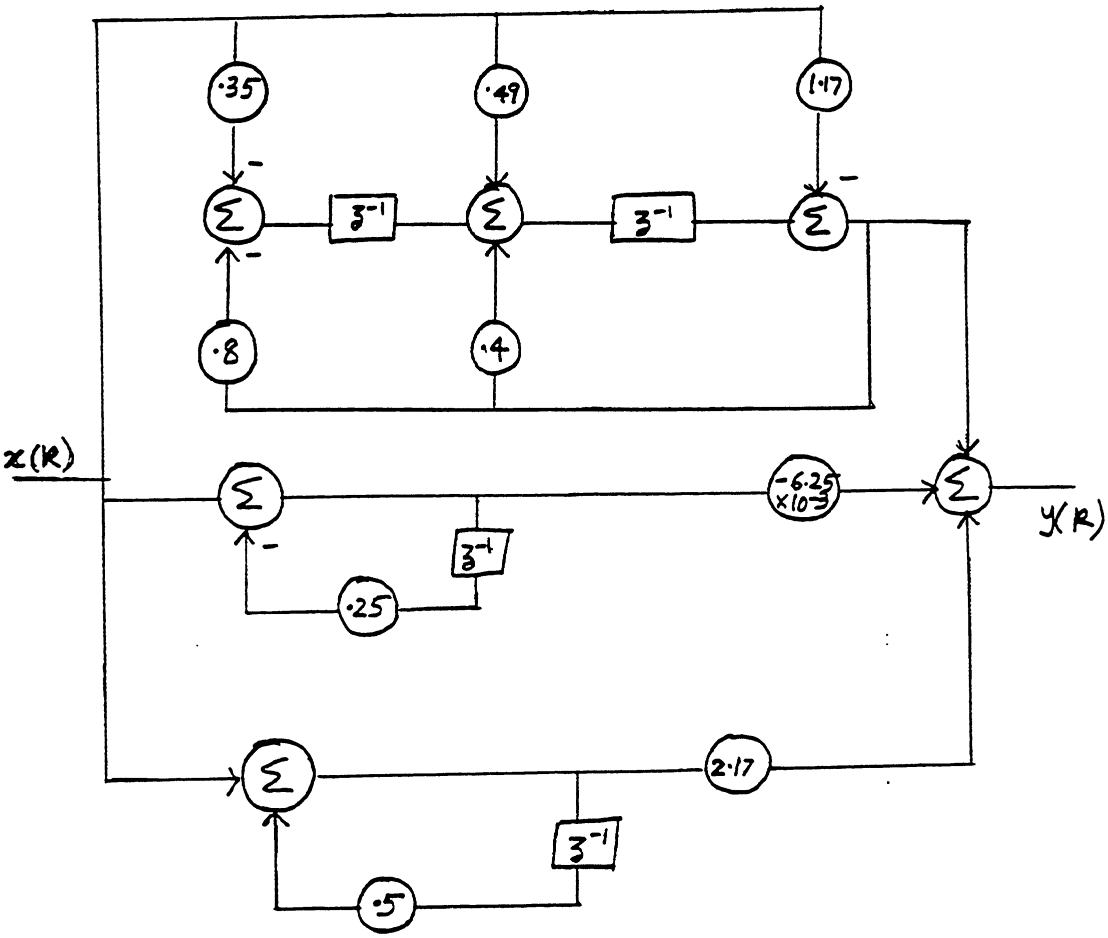
[///] – stable region

8.16 (i) $$H(z) = \frac{(1 + .3z^{-1})(1 + .2z^{-1})(1 + .8z^{-1})(1 + .9z^{-1})}{(1 + .4z^{-1} + 0.8z^{-2})\ (1 + 0.25z^{-1})(1 - .5z^{-1})}$$

$$= \frac{1 + .5z^{-1} + .06z^{-2}}{1 + .4z^{-1} + .8z^{-2}} \cdot \frac{1 + .8z^{-1}}{1 + .25z^{-1}} \cdot \frac{1 + .9z^{-1}}{1 - .5z^{-1}}$$



(ii) $$H(z) = \frac{-1.17 + .49z^{-1} - .35z^{-2}}{1 + .4z^{-1} + .8z^{-2}} + \frac{-6.25 \times 10^{-3}}{1 + .25z^{-1}} + \frac{2.17}{1 - .5z^{-1}}$$



8.17 Use 2nd Canonical form in all cases

(i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{24} & \frac{-9}{24} & \frac{13}{12} \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $c = [1 \ 0 \ 0]$ $d = 0$

(a) $\phi(k) = Z^{-1}\left[z(zI - A)^{-1} \right]$

$$= Z^{-1}\begin{bmatrix} \frac{\frac{28}{3}z}{z+\frac{1}{2}} - \frac{\frac{61}{7}z}{z+\frac{1}{3}} + \frac{\frac{8}{21}z}{z-\frac{1}{4}} & \frac{-7z/3}{z+\frac{1}{2}} + \frac{46z/21}{z+\frac{1}{3}} + \frac{\frac{35}{84}z}{z-1/4} & \frac{-4z}{z+\frac{1}{2}} + \frac{\frac{24}{7}z}{z+\frac{1}{3}} + \frac{\frac{12}{27}z}{z-\frac{1}{4}} \\ \frac{-z/6}{z+\frac{1}{2}} + \frac{z/7}{z+\frac{1}{3}} + \frac{z/42}{z-1/4} & \frac{-7z/6}{z+\frac{1}{2}} + \frac{\frac{-6z}{7}}{z+\frac{1}{3}} + \frac{\frac{5z}{42}}{z-1/4} & \frac{2z}{z+\frac{1}{2}} + \frac{\frac{-8z}{7}}{z+\frac{1}{3}} + \frac{\frac{1}{7}z}{z-1/4} \\ \frac{\frac{z}{12}}{z+\frac{1}{2}} + \frac{-z/21}{z+1/3} + \frac{\frac{1}{168}z}{z-\frac{1}{4}} & \frac{\frac{-11z}{12}}{z+\frac{1}{2}} + \frac{\frac{4}{7}z}{z+\frac{1}{3}} + \frac{\frac{-5z}{168}}{z-1/4} & \frac{z}{z+\frac{1}{2}} + \frac{8z/21}{z+\frac{1}{3}} + \frac{\frac{1}{28}z}{z-1/4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{28}{3}\left(-\frac{1}{2}\right)^k - \frac{61}{7}\left(-\frac{1}{3}\right)^k + \frac{8}{21}\left(\frac{1}{4}\right)^k & -\frac{7}{3}\left(\frac{-1}{2}\right)^k + \frac{46}{21}\left(\frac{-1}{3}\right)^k + \frac{35}{84}\left(\frac{1}{4}\right)^k & -4\left(\frac{-1}{2}\right)^k + \frac{24}{7}\left(\frac{-1}{3}\right)^k + \frac{12}{27}\left(\frac{1}{4}\right)^k \\ -\frac{1}{6}\left(-\frac{1}{2}\right)^k + \frac{1}{7}\left(\frac{-1}{3}\right)^k + \frac{1}{42}\left(\frac{1}{4}\right)^k & -\frac{7}{6}\left(\frac{-1}{2}\right)^k - \frac{6}{7}\left(\frac{-1}{3}\right)^k + \frac{5}{42}\left(\frac{1}{4}\right)^k & -2\left(\frac{-1}{2}\right)^k - \frac{8}{7}\left(\frac{-1}{3}\right)^k + \frac{1}{7}\left(\frac{1}{4}\right)^k \\ \frac{1}{12}\left(\frac{-1}{2}\right)^k - \frac{1}{21}\left(\frac{-1}{3}\right)^k + \frac{1}{168}\left(\frac{1}{4}\right)^k & -\frac{11}{12}\left(\frac{-1}{2}\right)^k + \frac{4}{7}\left(\frac{-1}{3}\right)^k - \frac{5}{168}\left(\frac{1}{4}\right)^k & \left(-\frac{1}{2}\right)^k + \frac{8}{21}\left(\frac{-1}{3}\right)^k + \frac{1}{28}\left(\frac{1}{4}\right)^k \end{bmatrix}$$

(b) With $X(z) = \frac{z}{z-1}$, $\underline{v}(0) = 0$

$$\underline{V}(z) = [zI - A]^{-1} b X(z) = \frac{1}{(z+\frac{1}{2})(z+\frac{1}{3})(z-\frac{1}{4})}\begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix} \frac{z}{z-1}$$

$$= \begin{bmatrix} \frac{\frac{2}{3}z}{z-1} + \frac{\frac{-11}{3}z}{z+\frac{1}{2}} + \frac{\frac{54}{7}z}{z+\frac{1}{3}} + \frac{-\frac{64}{21}z}{z-\frac{1}{4}} \\ \frac{\frac{2}{3}z}{z-1} + \frac{\frac{8}{3}z}{z+1/2} - \frac{\frac{18}{7}z}{z+1/3} - \frac{\frac{16}{21}z}{z-1/4} \\ \frac{\frac{2}{3}z}{z-1} - \frac{\frac{4}{3}z}{z+1/2} + \frac{\frac{6}{7}z}{z+\frac{1}{3}} - \frac{\frac{4}{21}z}{z-1/4} \end{bmatrix}$$

$$\therefore \underline{v}(k) = \begin{bmatrix} \frac{2}{3} - \frac{16}{3}\left(-\frac{1}{2}\right)^k + \frac{54}{7}\left(-\frac{1}{3}\right)^k - \frac{64}{21}\left(\frac{1}{4}\right)^k \\ \frac{2}{3} + \frac{8}{3}\left(-\frac{1}{2}\right)^k - \frac{18}{7}\left(-\frac{1}{3}\right)^k - \frac{16}{21}\left(\frac{1}{4}\right)^k \\ \frac{2}{3} - \frac{4}{3}\left(-\frac{1}{2}\right)^k + \frac{6}{7}\left(-\frac{1}{3}\right)^k - \frac{4}{21}\left(\frac{1}{4}\right)^k \end{bmatrix}$$

$$y(k) = v_1(k)$$

(c) $H(z) = c\,[z\,I - A]^{-1}\,b$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} [z\ I - A]^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\left(z + \frac{1}{2}\right)\left(z + \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

(ii) $A = \begin{bmatrix} 0 & 1 \\ -1 & -.707 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} -.207 & -0.75 \end{bmatrix}$

$d = 1$

(a)

$$\phi(k) = Z^{-1} \begin{bmatrix} \frac{z(z + .707)}{z^2 + .707z + 1} & \frac{z}{z^2 + .707z + 1} \\ \frac{-z}{z^2 + .707z + 1} & \frac{z^2}{z^2 + .707z + 1} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(1.932k) + .378\sin(1.932k) & 1.069\sin(1.932k) \\ -1.069\sin(1.932k) & \cos(1.932k) - .378\sin(1.932k) \end{bmatrix}$$

(b) $V(z) = \begin{bmatrix} \dfrac{z^2}{(z-1)(z^2+.707z+1)} \\ \\ \dfrac{z^3}{(z-1)(z^2+.707z+1)} \end{bmatrix} = \begin{bmatrix} \dfrac{Az+B}{z^2+.707z+1} + \dfrac{C}{z-1} \\ \\ \dfrac{-Bz+C}{z^2+.707z+1} + \dfrac{C}{z-1} \end{bmatrix}$

where $A = -.3693$, $B = -.6306$, $C = .3693$

$$\therefore V(k) = \begin{bmatrix} -.3693 \cos(1.932k) - .5238 \sin(1.932k) + .3693 \\ -.6306 \cos(1.932k) + .6304 \sin(1.932k) + .3693 \end{bmatrix}$$

and $y(k) = -.207\, V_1(k) - 0.75\, V_2(k) + 1$

(c) $H(z) = c\,(zI - A)^{-1} b = \begin{bmatrix} \frac{1}{2} - \frac{1}{\sqrt{2}} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} \dfrac{1}{z^2+.707z+1} \\ \\ \dfrac{z}{z^2+.707z+1} \end{bmatrix} + 1$

$$= \frac{z^2 + \frac{1}{2}z + \frac{1}{4}}{z^2 + .707z + 1}$$

(iii) (a) $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad d = 0$

$$\phi(k) = Z^{-1} \begin{bmatrix} \dfrac{2z}{z-1} - \dfrac{z}{z-2} & \dfrac{-z}{z-1} + \dfrac{z}{z-2} \\ \\ \dfrac{+2z}{z-1} - \dfrac{2z}{z-2} & \dfrac{-z}{z-1} + \dfrac{2z}{z-2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 - (2)^k & -1 + (2)^k \\ \\ 2 - 2(2)^k & -1 + 2(2)^k \end{bmatrix}$$

(b) $$V(z) = \begin{bmatrix} \dfrac{-z}{(z-1)^2} + \dfrac{z}{(z-1)(z-2)} \\[2ex] \dfrac{-z}{(z-1)^2} + \dfrac{2z}{(z-1)(z-2)} \end{bmatrix}$$

$$\therefore\ v(k) = \begin{bmatrix} -k-1 + (2)^k \\ -k-2 + 2(2)^k \end{bmatrix}$$

$$y(k) = -k-1 + (2)^k$$

8.18 (i) $$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\phi(n) = Z^{-1}\begin{bmatrix} \dfrac{z+1}{z^2+z+1} & \dfrac{1}{z^2+z+1} \\[2ex] \dfrac{-1}{z^2+z+1} & \dfrac{z}{z^2+z+1} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\cos \frac{2\pi}{3}(k-1) + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3}(k-1)\right] u(k-1) & \frac{2}{\sqrt{3}} \sin \frac{2\pi}{3}(k-1)\, u(k-1) \\[2ex] -\frac{2}{\sqrt{3}} \sin\left(\frac{2\pi}{3}(k-1)\right) u(k-1) & \frac{2}{\sqrt{3}} \sin \frac{2\pi}{3} k\ u(k) \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \dfrac{z}{(z^2+z+1)(z-1)} \\[2ex] \dfrac{z^2}{(z^2+z+1)(z-1)} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{1}{3}z}{z-1} + \dfrac{-\frac{1}{3}z^2 - \frac{2}{3}z}{z^2+z+1} \\[2ex] \dfrac{\frac{1}{3}z}{z-1} - \dfrac{1}{3}\,\dfrac{z^2-1}{z^2+z+1} \end{bmatrix}$$

$$v(k) = \begin{bmatrix} \frac{1}{3} - \frac{1}{3} \cos \frac{2\pi}{3} k - \sqrt{3} \sin \frac{2\pi}{3} k \\[1ex] \frac{1}{3} - \frac{1}{3} \cos \frac{2\pi}{3} k + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3} k \end{bmatrix} \qquad y(k) = v_1(k)$$

(ii) $A = \begin{bmatrix} 0 & 1 \\ 1/8 & 1/4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} -1/2 & 1 \end{bmatrix}$

$$\phi(n) = Z^{-1} \begin{bmatrix} \dfrac{z(z - \frac{1}{4})}{(z - 1/2)(z + 1/4)} & \dfrac{z}{(z - 1/2)(z + 1/4)} \\[2ex] \dfrac{1/8}{(z - 1/2)(z + 1/4)} & \dfrac{z^2}{(z - 1/2)(z + 1/4)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}\left(\frac{1}{2}\right)^k + \frac{1}{3}\left(-\frac{1}{4}\right)^k & \frac{4}{3}\left(\frac{1}{2}\right)^k - \frac{4}{3}\left(-\frac{1}{4}\right)^k \\[1ex] \frac{1}{6}\left(\frac{1}{2}\right)^k - \frac{4}{6}\left(-\frac{1}{4}\right)^k & \frac{2}{3}\left(\frac{1}{2}\right)^k + \frac{1}{3}\left(-\frac{1}{4}\right)^k \end{bmatrix}$$

$$V(z) = Z^{-1} \begin{bmatrix} \dfrac{z^2}{(z - \frac{1}{2})(z + \frac{1}{4})(z - 1)} \\[2ex] \dfrac{z^3}{(z - \frac{1}{2})(z + \frac{1}{4})(z - 1)} \end{bmatrix} = \begin{bmatrix} \dfrac{-\frac{4}{3} z}{z - \frac{1}{2}} \quad \dfrac{\frac{4}{15} z}{z + \frac{1}{4}} - \dfrac{\frac{8}{5} z}{z - 1} \\[2ex] \dfrac{-\frac{2}{3} z}{z - \frac{1}{2}} + \dfrac{\frac{1}{15} z}{z + 1/4} + \dfrac{\frac{24}{15} z}{z - 1} \end{bmatrix}$$

$$V(k) = \begin{bmatrix} -\frac{4}{3}\left(\frac{1}{2}\right)^k - \frac{4}{15}\left(-\frac{1}{4}\right)^k - 1 \\[1ex] -\frac{2}{3}\left(\frac{1}{2}\right)^k + \frac{1}{15}\left(-\frac{1}{4}\right)^k + \frac{24}{15} \end{bmatrix}$$

$$y(k) = -\frac{1}{2} V_1(k) + V_2(k) = \frac{1}{5}\left(-\frac{1}{4}\right)^k + \frac{21}{10}$$

(iii) $A = \begin{bmatrix} 0 & 1 \\ -\frac{15}{64} & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\phi(n) = Z^{-1} \begin{bmatrix} \dfrac{z(z - 1)}{(z - \frac{5}{8})(z - \frac{3}{8})} & \dfrac{z}{(z - \frac{5}{8})(z - \frac{3}{8})} \\[2ex] \dfrac{-15/64\, z}{(z - \frac{5}{8})(z - \frac{3}{8})} & \dfrac{z^2}{(z - 5/8)(z - 3/8)} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2}\left(\frac{5}{8}\right)^k + \frac{5}{2}\left(\frac{3}{8}\right)^k & 4\left(\frac{5}{8}\right)^k - 4\left(\frac{3}{8}\right)^k \\ -\frac{15}{16}\left(\frac{5}{8}\right)^k + \frac{15}{16}\left(\frac{3}{8}\right)^k & \frac{5}{2}\left(\frac{5}{8}\right)^k - \frac{3}{2}\left(\frac{3}{8}\right)^k \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \dfrac{z}{(z-\frac{5}{8})(z-\frac{3}{8})(z-1)} \\ \dfrac{z^2}{(z-\frac{5}{8})(z-\frac{3}{8})(z-1)} \end{bmatrix} = \begin{bmatrix} \dfrac{-\frac{32}{3} z}{z - 5/8} + \dfrac{\frac{32}{5} z}{z - 3/8} + \dfrac{\frac{64}{15} z}{z-1} \\ \dfrac{-\frac{20}{3} z}{z - 5/8} + \dfrac{\frac{12}{5} z}{z - 3/8} + \dfrac{\frac{64}{15} z}{z-1} \end{bmatrix}$$

$$V(k) = \begin{bmatrix} -\frac{32}{3}\left(\frac{5}{8}\right)^k + \frac{32}{5}\left(\frac{3}{8}\right)^k + \frac{64}{15} \\ -\frac{20}{3}\left(\frac{5}{3}\right)^k + \frac{12}{5}\left(\frac{3}{8}\right)^k + \frac{64}{15} \end{bmatrix}$$

$$y(k) = V_1(k)$$

(iv) $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{9} & -\frac{2}{3} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\Phi(n) = Z^{-1} \begin{bmatrix} \dfrac{z(z+\frac{2}{3})}{(z+\frac{1}{3})^2} & \dfrac{-z}{(z+\frac{1}{3})^2} \\ \dfrac{\frac{1}{9} z}{(z+\frac{1}{3})^2} & \dfrac{z^2}{(z+\frac{1}{3})^2} \end{bmatrix}$$

$$= \begin{bmatrix} (1-k)\left(-\frac{1}{3}\right)^k & 3k\left(-\frac{1}{3}\right)^k \\ -\frac{1}{3}k\left(-\frac{1}{3}\right)^k & (k+1)\left(-\frac{1}{3}\right)^k \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \dfrac{z}{(z-1)(z+\frac{1}{3})^2} \\ \dfrac{z^2}{(z-1)(z+\frac{1}{3})^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{9}{16}z}{z-1} - \dfrac{\frac{9}{16}z^2 + \frac{15}{16}z}{(z+\frac{1}{3})^2} \\ \dfrac{\frac{9}{16}z}{z-1} - \dfrac{\frac{9}{16}z^2 + \frac{1}{16}z}{(z+\frac{1}{3})^2} \end{bmatrix}$$

$$V(n) = \begin{bmatrix} \frac{9}{16} + \frac{9}{4}k\left(-\frac{1}{3}\right)^k - \frac{9}{16}\left(-\frac{1}{3}\right)^k \\ \frac{9}{16} - \frac{3}{8}k\left(-\frac{1}{3}\right)^k - \frac{9}{16}\left(-\frac{1}{3}\right)^k \end{bmatrix}$$

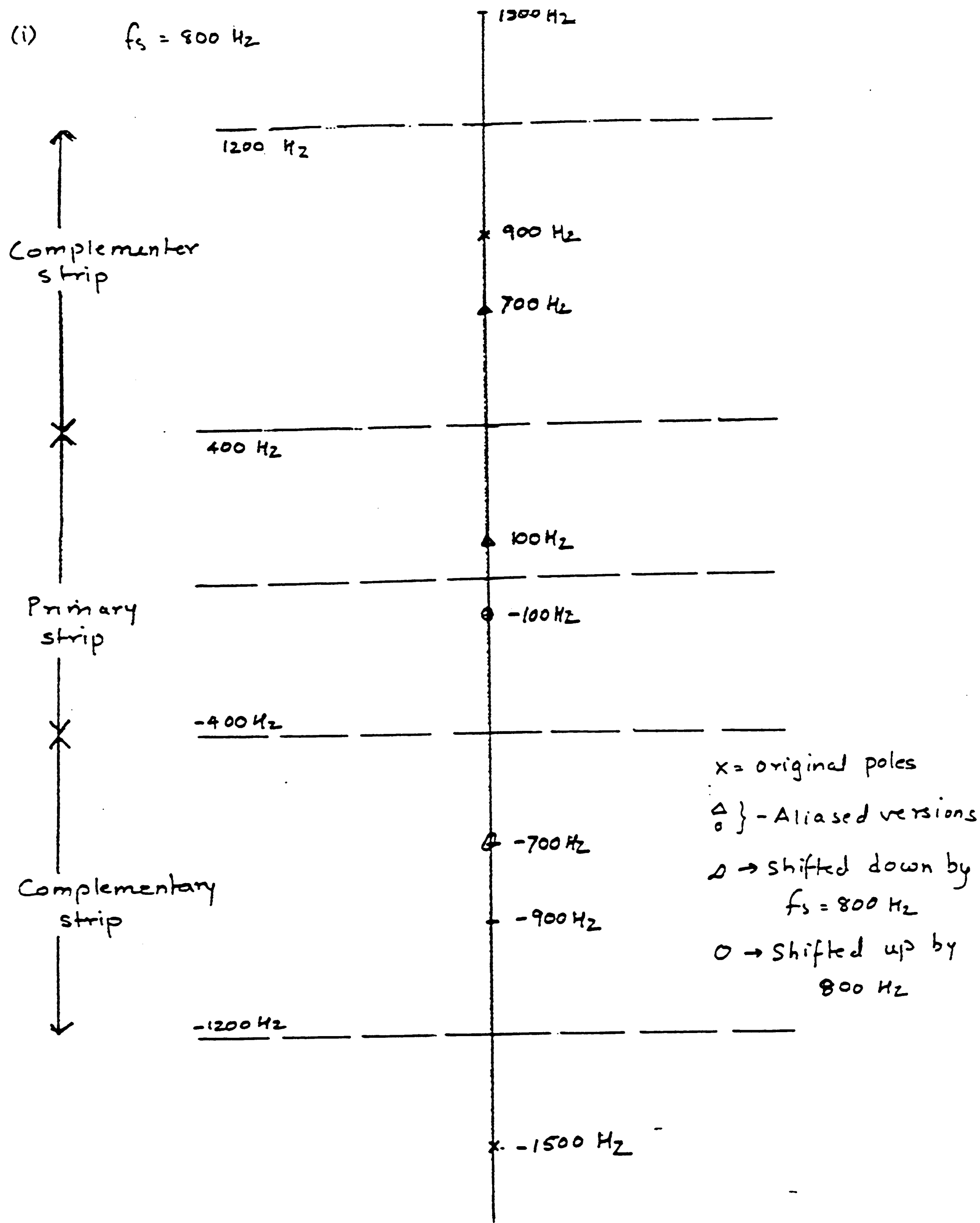
$$y(k) = v_1(k)$$

(v) Not a proper transfer function

8.19 (a) $f_s \geq 6\,kHz$, $T \leq \frac{1}{6000}$ secs

(b) Range of frequencies mapped into unit circle is $\left[-\frac{f_s}{2}, \frac{f_s}{2}\right] = [-3\,kHz, 3\,kHz]$

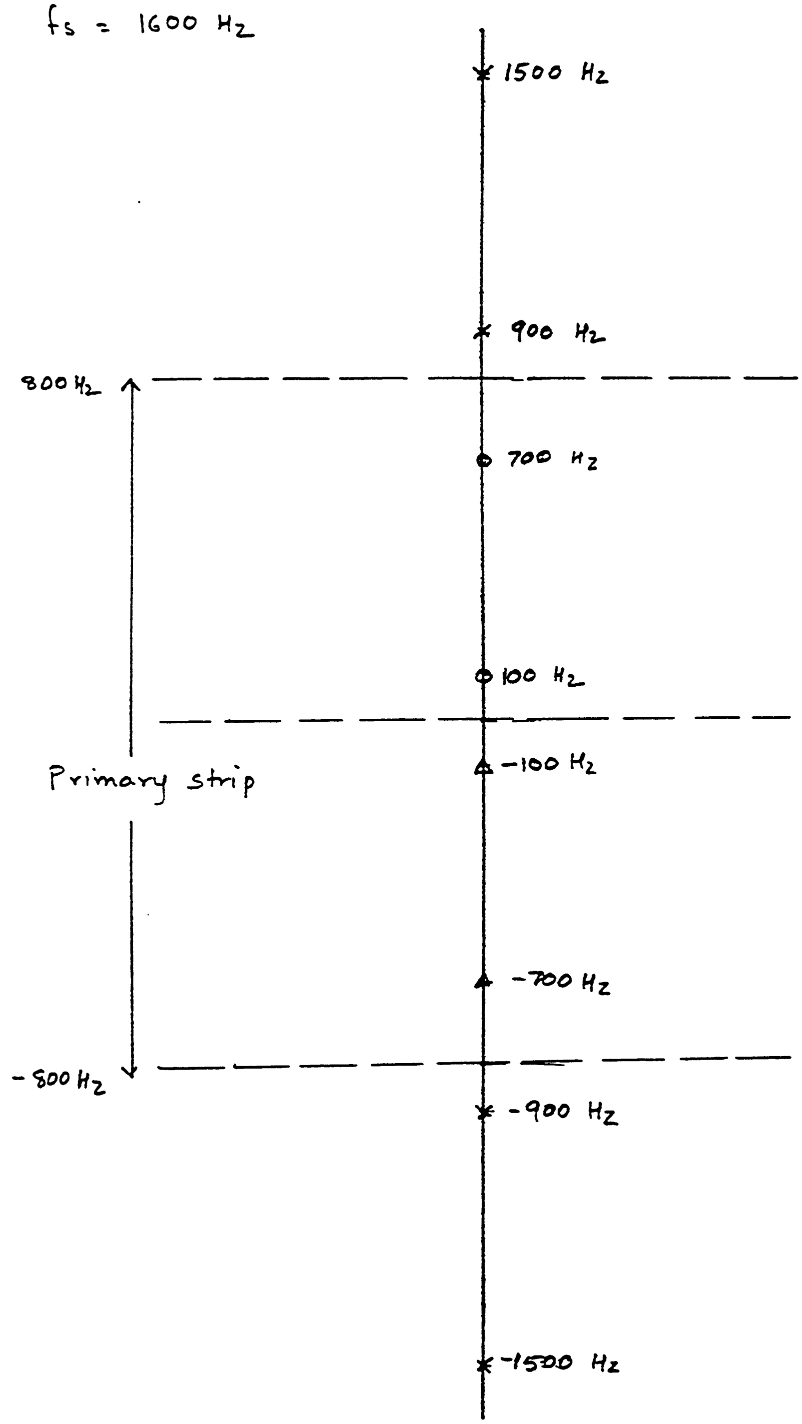
8.20 (i) $f_s = 800$ Hz



If we use a l.p. filter which passes frequencies only up to 1500 Hz, output will have components at

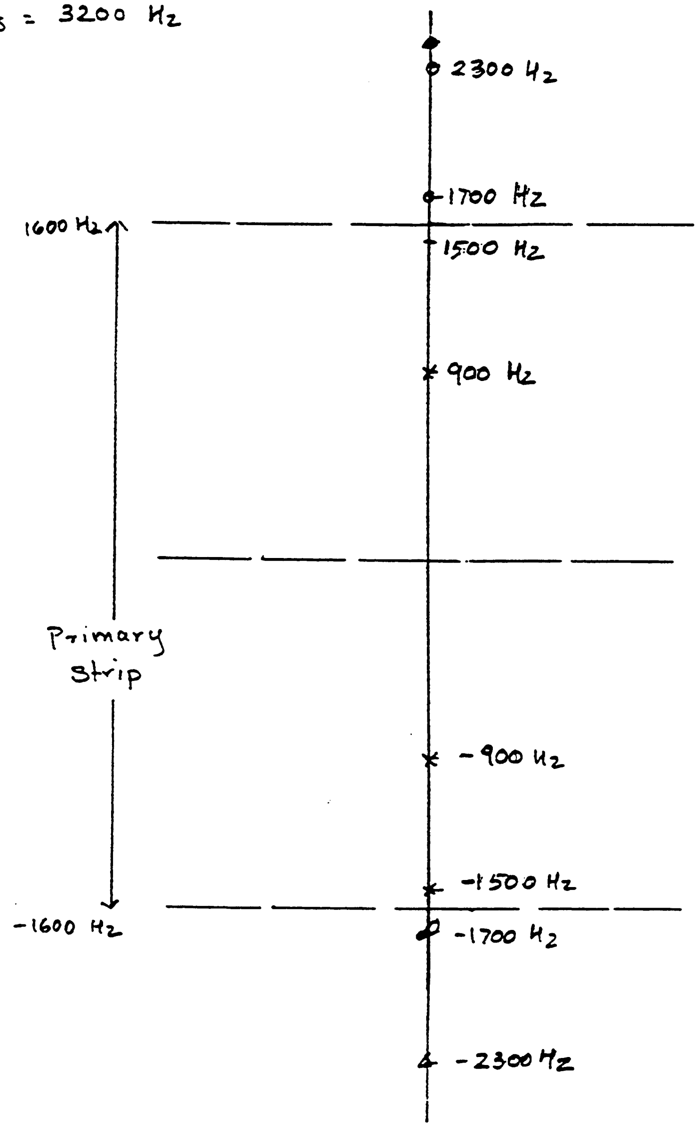
$\underbrace{100,\ 700}_{\text{Distortion}},\ \underbrace{900 \text{ and } 1500 \text{ Hz}}_{\text{Desired Components}}$

(ii) $f_s = 1600$ Hz



L.P filter with cutoff 1500 Hz will give an output with components at 100 Hz, 700 Hz, 900 Hz, 1500 Hz

(iii) $f_s = 3200$ Hz



LPF at 1500 Hz passes only 900 Hz & 1500 Hz components. No aliasing.

8.21 To find the impulse response, set $x_a(0) = 1$, all other $x(nT) = 0$

∴ For $0 \le t < T$, $h(t) = 1 + \frac{t}{T}$

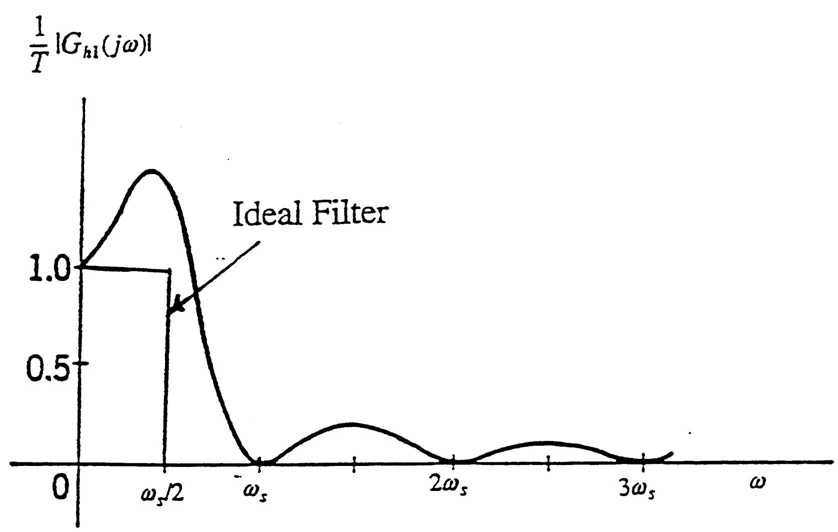
For $T \le t < 2T$, $h(t) = \frac{t - T}{T}(-1) = 1 - \frac{t}{T}$

$$H(s) = \int_0^T \left(1 + \frac{t}{T}\right) e^{-st}\,dt + \int_T^{2T} \left(1 - \frac{t}{T}\right) e^{-st}\,dt$$

$$= \int_0^{2T} e^{-st}\,dt + \frac{1}{T}\int_0^T t\,e^{-st}\,dt - \frac{1}{T}\int_T^{2T} t\,e^{-st}\,dt$$

$$= \frac{\left[1 - e^{-Ts}\right]^2 (1 + sT)}{T s^2} = \frac{1 + Ts}{T}\left[\frac{1 - e^{-Ts}}{s}\right]^2$$

$$= G_{h1}(s)$$



8.22 (a) For the derivative operator $y(t) = \frac{d}{dt} x(t)$, we have $H(s) = s$

For the backward difference approximation,

$$Y(z) = T\,X(z) + z^{-1} Y(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{T} \quad \therefore s = \frac{1 - z^{-1}}{T}$$

For the trapezoidal approximation,

$$Y(z) = \frac{T}{2}\left[X(z) + z^{-1} X(z) \right] + z^{-1} Y(z)$$

or $$H(z) = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\therefore s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

(b) For the first mapping, $z = \frac{1}{1 - sT}$

$s = 0$ gives $z = 1$

For $s = j\omega$, $z = \frac{1}{1 - j\omega T} = \frac{1}{2}\left[1 + \frac{1 + j\omega T}{1 - j\omega T} \right]$

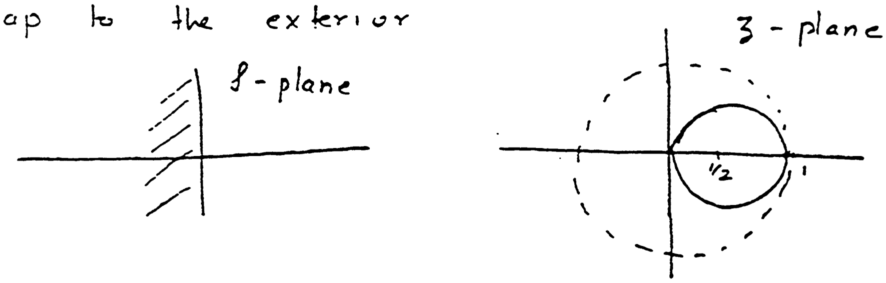
$$= \frac{1}{2}\left[1 + e^{j 2 \tan^{-1} \omega T} \right]$$

which is a circle with center $z = \frac{1}{2}$ and radius $\frac{1}{2}$

For $s = \sigma + j\omega$, $z = \frac{1}{2(1 - \sigma T)} \left[1 + e^{j 2 \tan^{-1} \omega T} \right]$

So that points along the line $\mathrm{Re}\, s = \sigma$ map into a circle with radius $\frac{1}{2(1 - \sigma T)}$ and center $z = \frac{1}{2}$

Thus points in the left-half s-plane map into the interior of the circle with center $z = \frac{1}{2}$, radius $= \frac{1}{2}$ while points in right half-plane map to the exterior



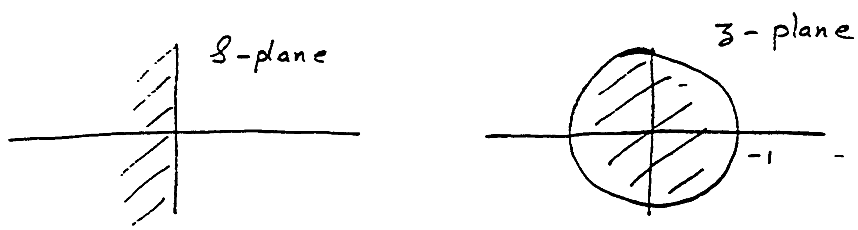
For $z = \dfrac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$, when $s = j\omega$, we have

$$z = \frac{1 + j\omega\frac{T}{2}}{1 - j\omega\frac{T}{2}} = e^{2j\tan^{-1}\frac{\omega T}{2}}$$

So that the imaginary axis of the s-plane maps into the unit circle in the z-plane.

For $s = \sigma$, we have $z = \dfrac{1 + \sigma T/2}{1 - \sigma T/2}$

So that $|z| < 1$ if $\sigma < 0$ and $|z| > 1$ if $\sigma > 0$
i.e The left-half s-plane maps into the interior of the unit circle and the right-half plane to the exterior



CHAPTER 9

9.1 (a) $X(k) = e^{-j\frac{2\pi}{N}kn_0} \qquad 0 \le k \le N-1$

(b) $X(k) = \sum_{n=0}^{N-1} (-1)^n e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} (e^{j\pi} e^{-j\frac{2\pi}{N}k})^n$

For N odd, $X(k) = \begin{cases} 2/(1+e^{-j\frac{2\pi}{N}k}) & k \neq 0 \\ 1 & k = 0 \end{cases}$

For N even, $X(k) = \begin{cases} N & k = N/2 \\ 0 & \text{otherwise} \end{cases}$

(c) We can assume N is even. Then

$X(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = \sum_{m=0}^{\frac{N}{2}-1} e^{-j\frac{4\pi}{N}mk}$

$= \begin{cases} \frac{N}{2} & k = 0, \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$

9.2 $X(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}n(N-k)}$

$= \left[\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} \right]^* = X^*(k)$

9.3 (i) $Y_1(k) = \sum_{\substack{n=0 \\ n \text{ even}}}^{2N-1} x(\frac{n}{2}) e^{-j\frac{2\pi}{N}kn} \qquad$ Let $m = \frac{n}{2}$

$Y_1(k) = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}km} = X(k)$

Note that $Y_1(k)$ is a periodic sequence with period 2N, while $X(k)$ is periodic with period N. That is, $Y_1(k)$ consists of two complete periods of $X(k)$

(ii) $Y_2(k) = \sum_{n=0}^{N-1} x(N-n-1) e^{-j\frac{2\pi}{N}kn} = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}k(N-n-1)}$

$= e^{j\frac{2\pi}{N}k} X(-k)$

(iii) $Y_3(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) e^{-j\frac{2\pi}{N/2}nk} = \sum_{m=0}^{N-1} x(m) \left[\frac{1+(-1)^m}{2} \right] e^{-j\frac{2\pi}{N/2}\frac{m}{2}k}$

$= \frac{1}{2} \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}mk} + \frac{1}{2} \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}(k+\frac{N}{2})}$

$= \frac{1}{2} X(k) + X(k+\frac{N}{2})$

(iv) $Y_4(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{2N}nk} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(\frac{k}{2})n}$

$= \begin{cases} X(\frac{k}{2}) & \text{for } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$

9.4 $$Y(k) = \sum_{n=0}^{16} x(n)W_{16}^{nk} = \sum_{n=0}^{7} x(n)W_{16}^{nk} + \sum_{n=8}^{15} x(n-8)W_{16}^{nk}$$

$$= \sum_{n=0}^{7} x(n)W_8^{n(\frac{k}{2})} + [\sum_{n=0}^{7} x(n-8)W_8^{n(\frac{k}{2})}]e^{-j\pi k}$$

$$= (1+e^{-j\pi k})X(\frac{k}{2}) = \begin{cases} 2X(\frac{k}{2}) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

so that

$$Y(0) = 2,\ Y(2) = Y^*(14) = 2 + j4,\ Y(4) = Y^*(6) = Y(10) = Y^*(12) = 2 - j2$$

$$Y(8) = 2 \text{ with all other } Y(k) = 0,\ 0 \le k \le 15.$$

9.5 (a)(i) $$X(k) = 1 - e^{-j\frac{\pi}{3}k} - e^{-j\frac{2\pi}{3}k} + e^{-j\pi k} - e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}$$

so that

$$X(0) = 0,\ X(1) = 1 + j\sqrt{3} = X^*(5),\ X(2) = 3 + j\sqrt{3} = X^*(4),\ X(3) = -2$$

Also

$$H(k) = 1 + 2e^{-j\frac{\pi}{3}k} + 3e^{-j\frac{2\pi}{3}k} + 3e^{-j\pi k} + 2e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}$$

which gives

$$H(0) = 12,\ H(1) = -3 - j\sqrt{3} = H^*(5),\ H(2) = H(3) = H(4) = 0$$

so that

$$Y(1) = -j4\sqrt{2} = Y^*(5),\ Y(0) = Y(2) = Y(3) = Y(4) = 0.$$

Hence

$$y_p(n) = \frac{1}{6}\sum_{k=0}^{5} Y(k)e^{j\frac{\pi}{3}kn}$$

$$= \{0, 2, 2, 0, -2, -2\}$$

9.5 (ii) $$X(k) = 1 - 2e^{-j\frac{\pi}{2}k} - e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

so that

$$X(0) = -1,\ X(1) = 2 + j3 = X^*(3),\ X(2) = 1$$

Since $H(k) = 1 + e^{-j\frac{3\pi}{2}k}$, we have

$$H(0) = 2,\ H(1) = 1 + j = H^*(3),\ H(2) = 0$$

and

$$Y(0) = -2,\ Y(1) = -1 + j5 = Y^*(3),\ Y(2) = 0$$

so that

$$y_p(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\frac{\pi}{2}kn}$$
$$= \{-1, -3, 0, 2\}$$

9.6 (a)(i) Zero-pad both sequences to length 11. Then, we have

$X(0) = 0$	$H(0) = 12$
$X(1) = X^*(10) = -0.7036 + j0.9345$	$H(1) = H^*(10) = 1.2326 - j8.5729$
$X(2) = X^*(9) = 1.2635 + j1.4979$	$H(2) = H^*(9) = -2.4556 - j0.7210$
$X(3) = X^*(8) = 1.0211 + j1.4026$	$H(3) = H^*(8) = 0.1108 - j0.2426$
$X(4) = X^*(7) = 4.0133 + j0.4167$	$H(4) = H^*(7) = 0.2835 + j0.1822$
$X(5) = X^*(6) = -0.0944 - j2.9141$	$H(5) = H^*(6) = 0.2328 - j0.3793$

$Y(k) = H(k)X(k)$ gives $Y(0) = 0$, $Y(1) = Y^*(10) = 7.1438 + j7.1838$

$Y(2) = Y^*(9) = -2.0227 - j4.5894$, $Y(3) = Y^*(8) = 0.4534 - j0.0923$,

$Y(4) = Y^*(7) = 1.062 + j0.8494$, $Y(5) = Y^*(6) = -1.1365 - j0.9221$

so that

$$y_l(n) = \frac{1}{11}\sum_{k=0}^{10} Y(k)e^{j\frac{2\pi}{11}kn}$$
$$= \{1, 1, 0, -1, -3, -2, -1, 1, 2, 1, 1\}$$

9.6 **(ii)** Zero-pad both sequences to length 7. Then

$X(0) = -1$ $\quad H(0) = 2$

$X(1) = 0.9254 + j2.1047 = X*(6) \quad H(1) = 0.099 - j4.3339 = H*(6)$

$X(2) = 2.9695 + j2.2978 = X*(5) \quad H(2) = 1.625 + j0.7818 = H*(5)$

$X(3) = 1.9559 - j0.88990 = X*(4) \quad H(3) = 0.7775 - j0.9749 = H*(4)$

$Y(k) = H(k)X(k)$ gives $Y(0) = -2$, $Y(1) = 0.8216 + j0.61 = Y*(6)$,

$Y(2) = 3.0245 + j6.0521 = Y*(5)$, $Y(3) = 0.654 - j0.25981 = Y(4)$

so that

$$y_l(n) = \frac{1}{7}\sum_{k=0}^{6} Y(k)e^{j\frac{2\pi}{7}kn}$$

$$= \{1, -2, -1, 2, -2, -1, 1\}$$

9.7 $X(\Omega) = \sum_{n=0}^{\infty} x(n)e^{-j\Omega n}$ $\quad$ Let $n = 8m+p$

$$X(\Omega) = \sum_{p=0}^{7}\sum_{m=0}^{8} x(8m+p)\,e^{-j\Omega(8m+p)}$$

$$X\left(\tfrac{2\pi}{8}k\right) = \sum_{p=0}^{7}\sum_{m=0}^{8} \left(\tfrac{1}{3}\right)^{8m+p} e^{-j\frac{2\pi}{8}k(8m+p)}$$

$$= \sum_{p=0}^{7}\left[\left(\sum_{m=0}^{8}\left(\tfrac{1}{3}\right)^{8m}\right)\left(\tfrac{1}{3}\right)^{p}\right]e^{-j\frac{2\pi}{8}kp}$$

$$= \sum_{p=0}^{7} \frac{1-\left(\frac{1}{3}\right)^{9}}{1-\frac{1}{3}}\left(\tfrac{1}{3}\right)^{p} e^{-j\frac{2\pi}{8}kp}$$

Let $\quad y(p) = \dfrac{1-(1/3)^{9}}{1-1/3}\left(\tfrac{1}{3}\right)^{p}$

Then $\quad X\left(\tfrac{2\pi}{8}k\right) = \sum_{p=0}^{7} y(p)e^{-j\frac{2\pi}{8}kp} = X(k)$

$k = 0, 1, \ldots 7$

9.8 $$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x^*(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}\right].$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x(n) W_N^{kn}\right]^* X(k)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) X^*(k)$$

9.9 (i) If $M > N$, augment $x(n)$ by adding $M - N$ zeros and take the M-point DFT.

(ii) If $M < N$, let $M = NL$ where L is an integer and let $n = mM + p$. Then

$$X(e^{j\frac{2\pi}{M}k}) = \sum_{n=0}^{ML} x(n) e^{-j\frac{2\pi}{M}k} = \sum_{p=0}^{M-1} \sum_{m=0}^{L-1} x(mM+p) e^{-j\frac{2\pi}{M}(mM+p)k}$$

$$= \sum_{p=0}^{M-1} \left[\sum_{m=0}^{L-1} x(mM+p)\right] e^{-j\frac{2\pi}{M}pk}$$

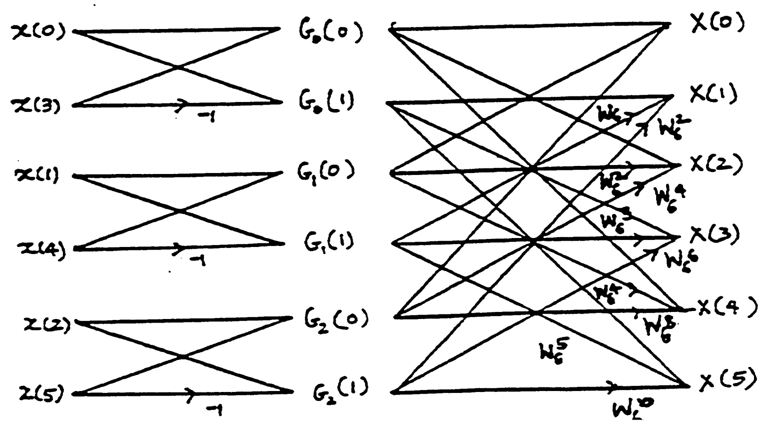
$$= M-\text{point DFT of } \sum_{m=0}^{L-1} x(mM+p)$$

9.10 128 samples in the range $[\frac{7\pi}{16}, \frac{15\pi}{16})$ correspond to 512 points in the range $[0, 2\pi)$. Hence do a 512-point DFT. Then $k = 112$ corresponds to $\Omega = 7\pi/16$ rads and $k = 239$ corresponds to $\Omega = (239\text{x}2\pi)/512 = [15\pi/16 - (2\pi)/512]$ rads. Use the procedure in Problem 9.9 to determine the 512-point DFT of $x(n)$ for the cases in (i) and (ii).

9.11 Use the results of Problem 9.9.

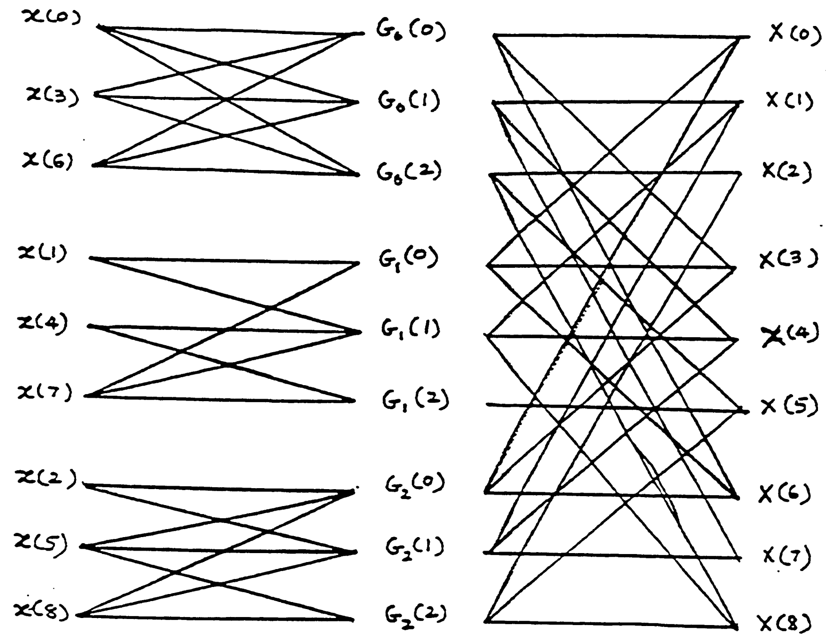
9.12 $$X(k) = \sum_{p=0}^{5} x(n) W_6^{nk} \qquad \text{Let } n = 3m+p$$

$$X(k) = \sum_{p=0}^{2} \left[\sum_{m=0}^{1} x(3m+p) W_2^{mk}\right] W_6^{pk} = \sum_{p=0}^{2} G_p(k) W_6^{pk}$$



9.13 $X(k) = \sum_{n=0}^{8} x(n) W_9^{nk}$

$$= \sum_{p=0}^{2} \left[\sum_{m=0}^{2} x(3m+p) W_3^{mk} \right] = \sum_{p=0}^{2} G_p^{k} W_9^{pk}$$



9.14 $\Delta f = \frac{1}{T_0} < 0.25, \quad T_0 > 4$ Sec

9.15 $T_0 = NT = 4096 \times \frac{1}{40{,}000}$

$$\Delta f = \frac{1}{T_0} = \frac{40{,}000}{4096} = 9.765 \text{ Hz}$$

9.16 (a) $T_0 = 24, \quad \Delta f = \frac{1}{T_0} = \frac{1}{24}$ Hz

(b) $N = \frac{T_0}{T} = f T_0 = \frac{128}{3} \times 24 = 1024$

$$\therefore \Delta\Omega = \frac{2\pi}{N} = \frac{2\pi}{1024} \text{ radians}$$

(c) For no aliasing, analog signal must be band limited to $21\frac{1}{3}$ Hz.

9.17 (a) $x(n) = \frac{1}{16}\left[2 + (4-j4)e^{j\frac{2\pi}{16}3n} - 2e^{j\frac{2\pi}{16}5n}\right.$

$\left. - \frac{1}{2}e^{j\frac{2\pi}{16}8n} - 2e^{j\frac{2\pi}{16}11n} + (4+j4)e^{j\frac{2\pi}{16}13n}\right]$

$= \frac{1}{16}\left[2 + 4\sqrt{2}\cos\left(\frac{2\pi}{16}3n - 45^\circ\right) - 2\cos\left(\frac{2\pi}{16}5n\right)\right.$

$\left. + \left(-\frac{1}{2}\right)^n\right]$

(b) $\Delta\Omega = \frac{2\pi}{16} = \frac{\pi}{8}$ rads.

(c) $T_0 = 4$ secs

(d) Highest frequency that does not cause aliasing $= 2$ Hz.

(e) $\Delta f = \frac{\Delta\Omega}{2\pi T} = \frac{1}{2}$ Hz ; $T_0 = \frac{1}{\Delta f} = 2$ sec.

9.18 (a) If $h(n)$ is real, $H^*(k) = H(-k)$

$\therefore H_R(k) = \frac{1}{2}H(k) + \frac{1}{2}H^*(k)$

$= \frac{1}{2}H(k) + \frac{1}{2}H(-k) = H_e(k)$

$= \frac{1}{2}H(k) + \frac{1}{2}H(N-k)$

$jH_I(k) = \frac{1}{2}H(k) - \frac{1}{2}H^*(k)$

$= \frac{1}{2}H(k) - \frac{1}{2}H(-k) = H_o(k)$

$= \frac{1}{2}H(k) - \frac{1}{2}H(N-k)$

(b) If $h(n)$ is purely imaginary, $H^*(k) = -H(-k)$

$\therefore H_R(k) = H_o(k) = \frac{1}{2}H(k) - \frac{1}{2}H(N-k)$

$jH_I(k) = H_e(k) = \frac{1}{2}H(k) + \frac{1}{2}H(N-k)$

(c) Let $x(n) = f(n) + jg(n) = f(n) + h(n)$

Then $X(k) = X_R(k) + jX_I(k)$

$$= X_{Re}(k) + X_{Ro}(k) + j[X_{Ie}(k) + X_{Io}(k)]$$

Also $X(k) = F(k) + H(k)$

$$= F_R(k) + jF_I(k) + H_R(k) + jH_I(k)$$

Since $f(n)$ is real, $F_R(k)$ is even, $jF_I(k)$ is odd.

Since $h(n)$ is imaginary, $H_R(k)$ is odd, $jH_I(k)$ is even

$\therefore F_R(k) = X_{Re}(k)$, $jF_I(k) = jX_{Io}(k)$

$H_R(k) = X_{Ro}(k)$, $jH_I(k) = jX_{Ie}(k)$

That is $F(k) = \dfrac{X_R(k) + X_R(N-k)}{2} + j\dfrac{X_I(k) - X_I(N-k)}{2}$

$$H(k) = jG(k) = \frac{X_R(k) - X_R(N-k)}{2} + j\frac{X_I(k) + X_I(N-k)}{2}$$

or $G(k) = \dfrac{X_I(k) - X_I(N-k)}{2} - j\dfrac{X_R(k) - X_R(N-k)}{2}$

9.19 The Fourier transform of signal $x_a(t)$ is given by

9.20

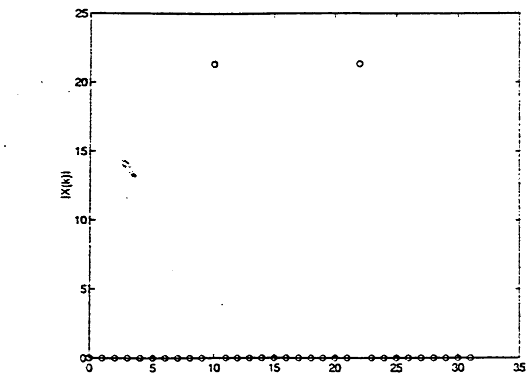
$$X_a(f) = 2[\delta(f - \frac{1}{3}) + \delta(f + \frac{1}{3})]$$

and consists of two δ-functions at $f = \pm 1/3$ Hz.

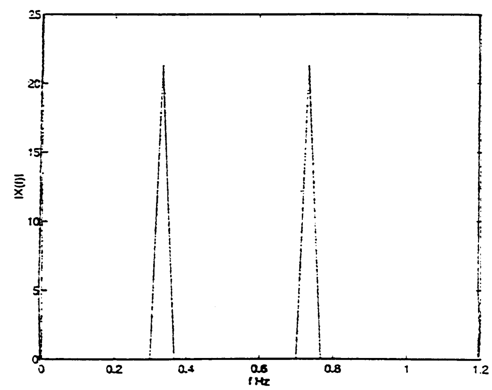
Figures (a)-(d) below show the results obtained using the DFT on a 32-point data sequence obtained by sampling $x_a(t)$ at intervals of $T = 15/16$ s, corresponding to a sampling frequency of $f_s = 16/15$ Hz. At this value of T, the duration of the analog signal used in computing the spectrum is 30 s. Figures (a) and (c) show plots of $|X(k)|$ versus k for the rectangular and Hamming windows respectively, while Figures (b) and (d) show plots of $|X(f)|$ versus f for $0 \leq f < f_s$ for the rectangular and Hamming windows.

Figures (e)-(h) below show the corresponding results obtained using the DFT when $x_a(t)$ is sampled at intervals of $T = 0.1$ s, corresponding to a sampling frequency of $f_s = 10$ Hz. The duration of the analog signal is 3.2 s or roughly 1/10th the duration in the previous case. Figures (e) and (g) show plots of $|X(k)|$ versus k for the recatangular and Hamming windows respectively, while Figures (f) and (h) show plots of $|X(f)|$ versus f for $0 \le f < f_s$ for the rectangular and Hamming windows.

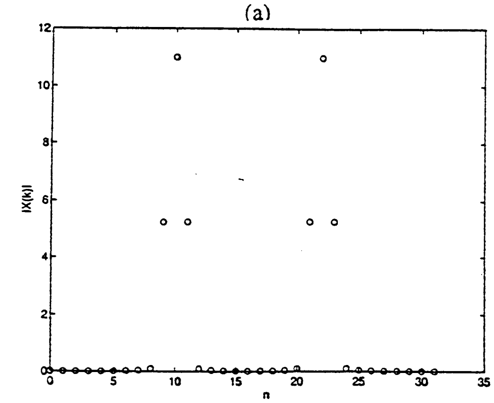
All the figures show two peaks, at frequencies of 1/3 Hz and $f_s - 1/3$ Hz.



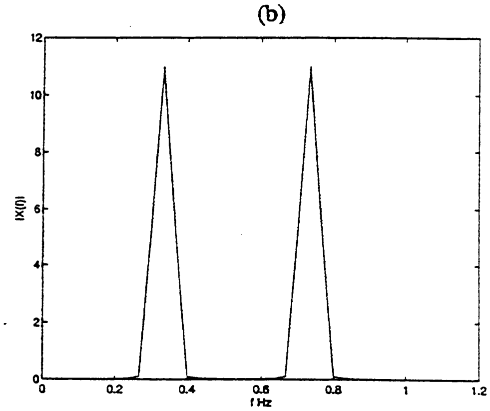
(a)



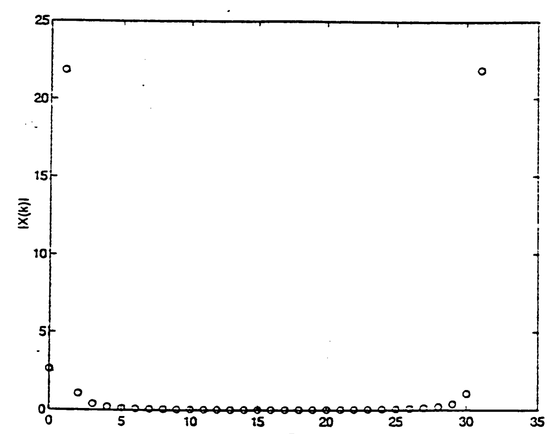
(b)



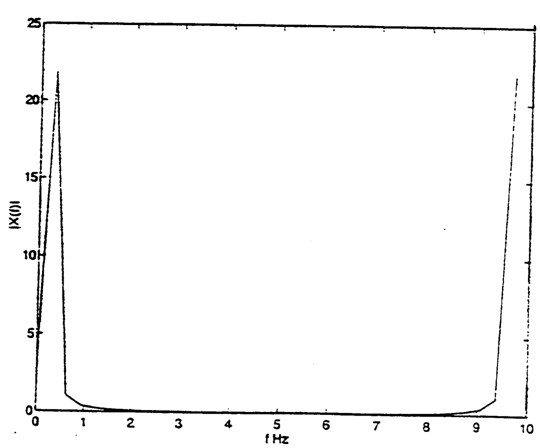
(c)



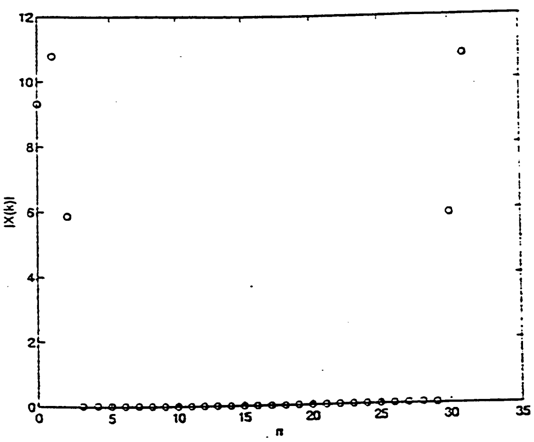
(d)



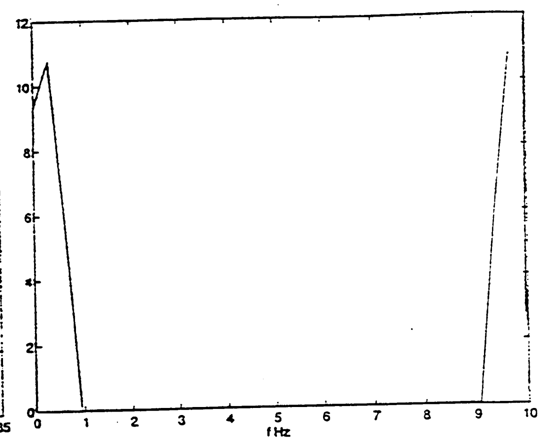
(e)



(f)



(g)



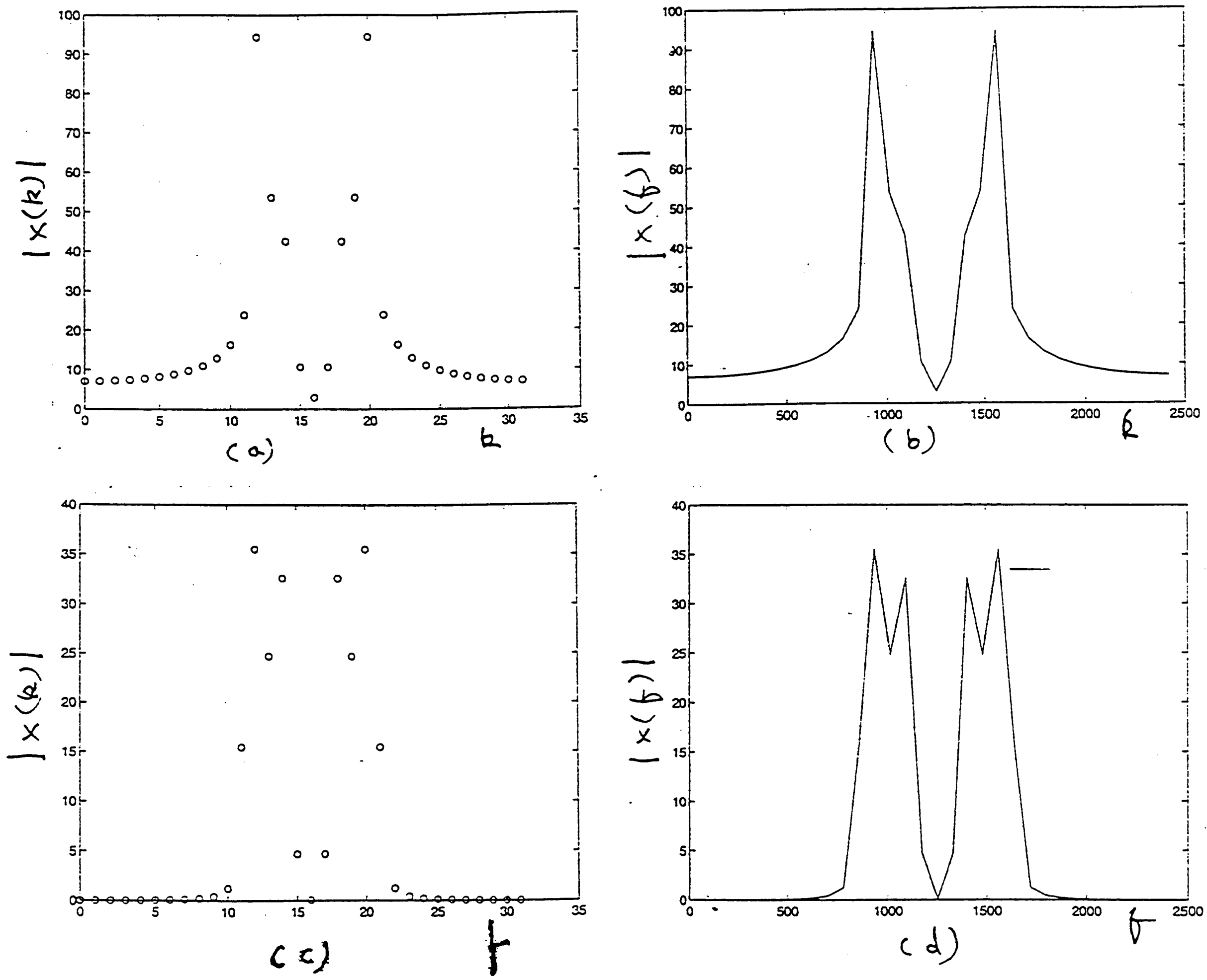
(h)

9.21

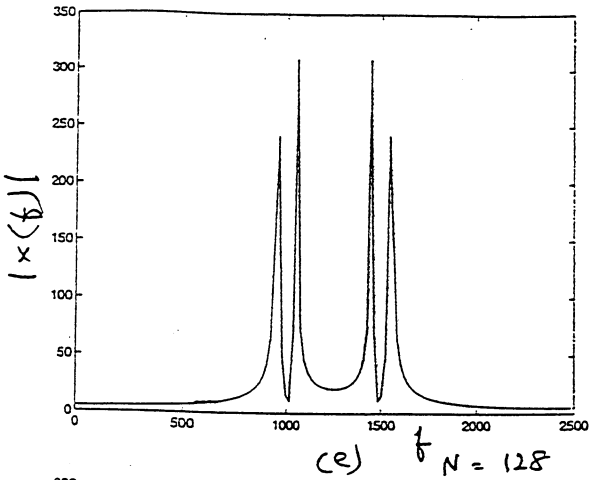
9.22 Since $x_a(t) = 5\cos(1900\pi t) + 5\cos(2100\pi t)$, $\Delta f = 100$ Hz. Thus $T_0 > 0.1$ sec. The sampling rate of $T = 0.4$ msec corresponds to a sampling frequency of $f_s = 2500$ Hz and satisfies the Nyquist criterion. With this value of T, the minimum number of samples required to resolve the two frequencies is $N = T_0/T = 25$. To use a radix-2 FFT algorithm, we therefore need at least 32 samples.

Figures (a) and (b) show the spectra obtained using a data length of 32 using a recatangular window. Figure (a) plots $|X(k)|$ vs k while (b) shows a plot of $|X_s(f)|$ vs f in the range $0 \le f < f_s$. Figures (c) and (d) show the corresponding results using a Hamming window. As can be seen from the plots, for a data length of $N = 32$, we are unable to resolve the two frequencies using a recatngualr window.

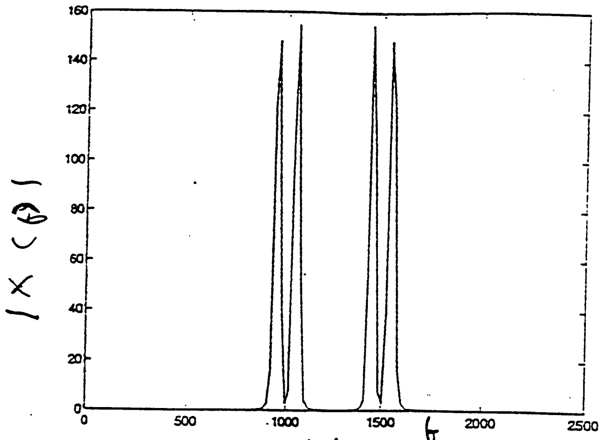
Figures (e) through (j) show corresponding results for data lengths of $N = 128$, 256 and 512 respectively. The figures on the left correspond to the recatangular window and those on the right to the Hamming window. For these data lengths, we can resolve the two frequencies with both windows. The peaks in the spectrum are narrower for the Hamming window. As the data length increases, the peaks become sharper.



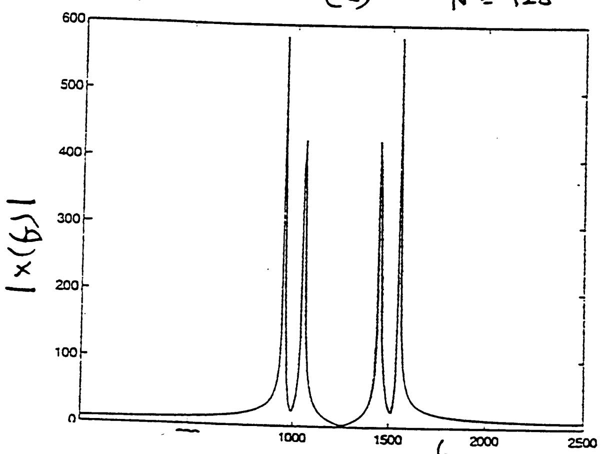
|X(k)|
(a)
k
|X(f)|
(b)
k
|X(k)|
(c)
f
|X(f)|
(d)
f



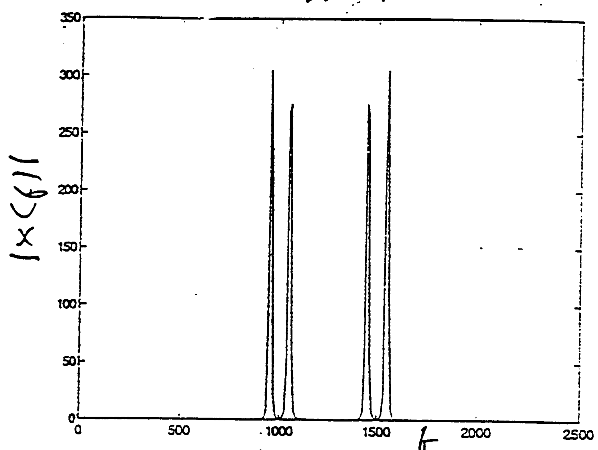
(e) N = 128



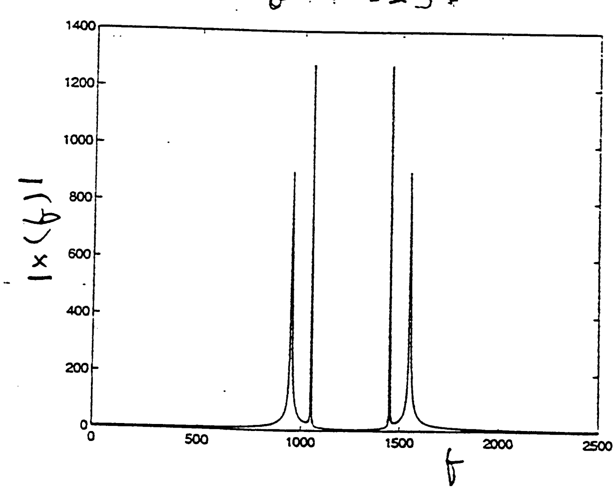
(f) N = 128



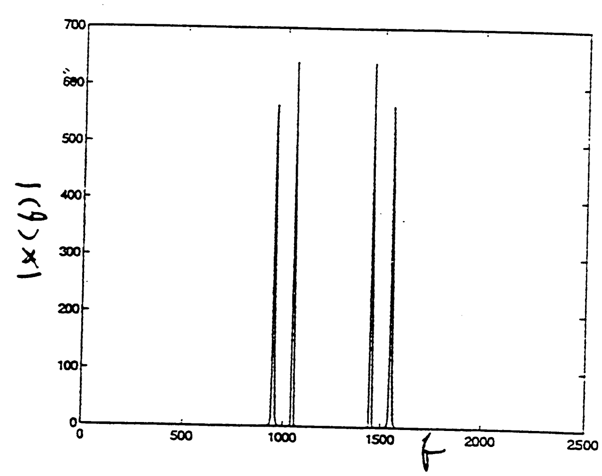
(g) N = 256



(h) N = 256



(i) N = 512



(j) N = 512

10.1 Set $20\log_{10}(1-\delta_1) = -1.5$ and $20\log_{10}\delta_2 = -15$ to get $\delta_1 = 0.1586$ and $\delta_2 = 0.1788$.

Equation (10.3.15) gives $N = 1.5535$. We choose $N = 3$. If we use Equation (10.3.13), we get $\omega_c = 1412.5$ rad/s. The unnormalized filter is

$$H(s) = \frac{(1412.5)^2}{s^2 + \sqrt{2}(1412.5)s + (1412.5)^2}$$

10.2 With $f_p = 3$ kHz and $N = 2$, we solve for ω_c as

$$\omega_c = \left[\frac{1-\delta_1}{\sqrt{2\delta_1 - \delta_1^2}}\right]^{\frac{1}{N}} (2\pi f^{p)} = 26622\text{rad/sec}$$

and

$$f_s = f_c\left[\frac{1}{\delta_2^2} - 1\right]^{\frac{1}{2N}} = 62631\text{rad/sec}$$

The filter transfer function is

$$H(s) = \frac{(26622)^2}{s^2 + \sqrt{2}(26622)s + (26622)^2}$$

10.3 We first design a normalized low pass filter with

$$20\log_{10}(1-\delta_1) = -1, \quad 20\log_{10}\delta_2 = -10$$

so that $\delta_1 = 0.108479$ and $\delta_2 = 0.31623$. Use of Eq. (10.13.15) gives $N = 1.012$. Set $N = 2$ to get

$$H(s) = \frac{1}{s^2 + 1.4142s + 1}$$

The required band pass filter is obtained by replacing by

$$\frac{\omega_0}{BW}\left(\frac{s^*}{\omega_0} + \frac{\omega_0}{s^*}\right) = \sqrt{6}\left(\frac{s^{*2} + 6\times10^4}{\sqrt{6}\;100s^*}\right)$$

$$\therefore H(s^*) = \frac{1}{\left(\frac{s^{*2} + 6\times10^4}{100s^*}\right)^2 + 1.414^2\left(\frac{s^{*2} + 6\times10^4}{100s^*}\right) + 1}$$

i.e. $H(s) = \dfrac{10^4 s^2}{s^4 + 141.42 s^3 + 12.01 \times 10^4 s^2 + 8.485 \times 10^6 s + 36 \times 10^{8}}$

by replacing s^{*} by s.

10.4 Normalize by setting $\omega_p = 1$ so that $\omega_s = \frac{10}{3}$

$$20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = -2 \quad \text{gives} \quad \epsilon = .7648$$

Also $20 = 20 \log_{10} (.7648) + 6(N-1) + 20 N \log_{10} \left(\frac{10}{3}\right)$

gives $N = 1.7213$

Choose $N = 2$. Also $\beta = \frac{1}{2} \sin^{-1} \frac{1}{.7648} = .5415$

$\therefore$ Poles of Chebyshev filter are at

$$s = \frac{-1500}{\sqrt{2}} \times .5684 \pm j \frac{1500}{\sqrt{2}} \times 1.1502$$

$$= -602.88 \pm j\, 1219.97$$

$$\therefore \quad H(s) = \frac{1}{(s + 602.88)^2 + (1219.97)^2}$$

10.5 The normalized Butterworth filter of order 3 is

$$H_B(s) = \frac{1}{(s^2+s+1)(s+1)} = \frac{1}{\left[(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2\right](s+1)}$$

To find the normalized Chebyshev filter, set

$$|H(1)| = \frac{1}{\sqrt{1+\epsilon^2}} = \frac{1}{\sqrt{2}} \quad \text{so that} \quad \epsilon = 1$$

So that $\beta = \frac{1}{3} \sinh^{-1}(1) = .2938$

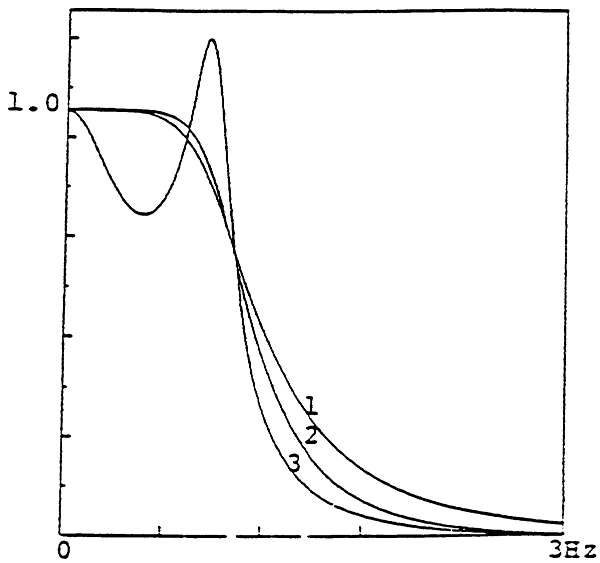
Complex roots of the Chebyshev filter are given by

$$s = -\frac{1}{2} \sinh(.2938) \pm j \frac{\sqrt{3}}{2} \cosh(.2938)$$

$$= -.149 \pm j\,.9036$$

Real root is $s = -\sinh(.2938) = .298$

$\therefore \quad H_c(s) = \dfrac{0.0723}{(s+.298)[(s+.149)^2+(.9036)^2]}$



10.6 From the above figure, need a 5^{th} order Butterworth filter.

10.7 From the specifications, $20\log_{10}\dfrac{1}{\sqrt{1+\varepsilon^2}} = -1.5$, so that $\varepsilon = 0.4825$.

Equation (10.3.22) gives

$$20\log_{10}(.4125) + 6(N-1) + 20N\log_{10}(5) = 15$$

from which $N = 1.0703$. We set $N = 2$, so that

$$\frac{s}{\omega_p} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

where $\omega_p = 2\pi 1000 = 6283.2$ rad/sec.

From Equation (10.3.26) $\beta = \dfrac{1}{2}\sinh^{-1}\dfrac{1}{.4125} = 1.6187$, so that the poles of $H(s)$ are at

$$s = -\frac{6283.2}{\sqrt{2}}\sinh(1.6187) \pm j\frac{6283.2}{\sqrt{2}}\cosh(1.6187)$$

$$= -1.077\text{x}10^5 \pm j1.1651\text{x}10^5$$

We therefore have

$$H(s) = \frac{1}{(s+1.077\text{x}10^5)^2 + (1.1651\text{x}10^5)^2}$$

10.8

$$H(s) = \frac{(1412.5)^2}{s^2 + \sqrt{2}(1412.5)s + (1412.5)^2}$$

From Table 10.5,

$$H(z) = \frac{ze^{-aT}\sin(\omega_0 T)}{z^2 - 2ze^{-aT}\cos(\omega_0 T) + e^{-2aT}}$$

where $\omega_0 = a = \dfrac{\omega_c}{\sqrt{2}} = 998.78$.

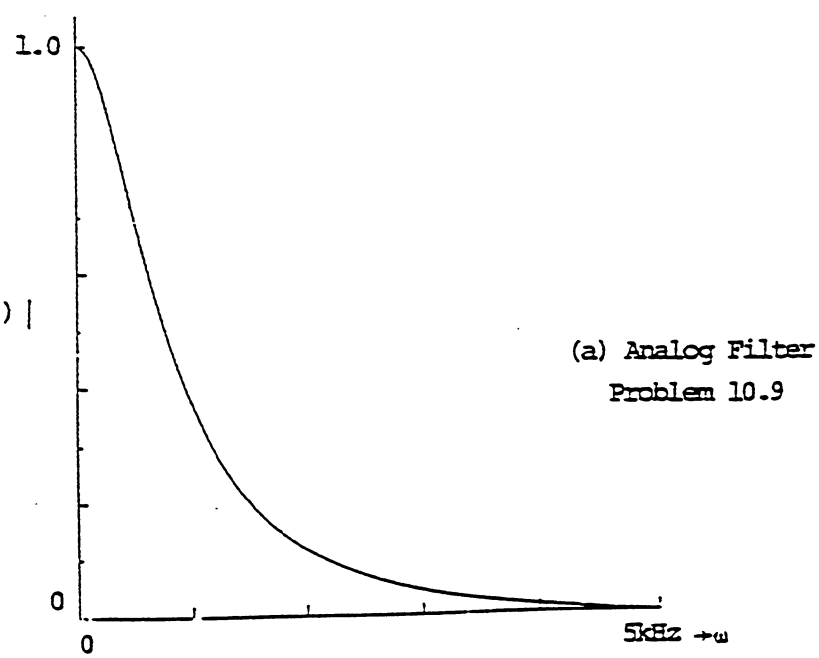
With $T = \dfrac{1}{6000}$,

$$H(z) = \frac{.403z}{z^2 - 1.67z + .7168}$$

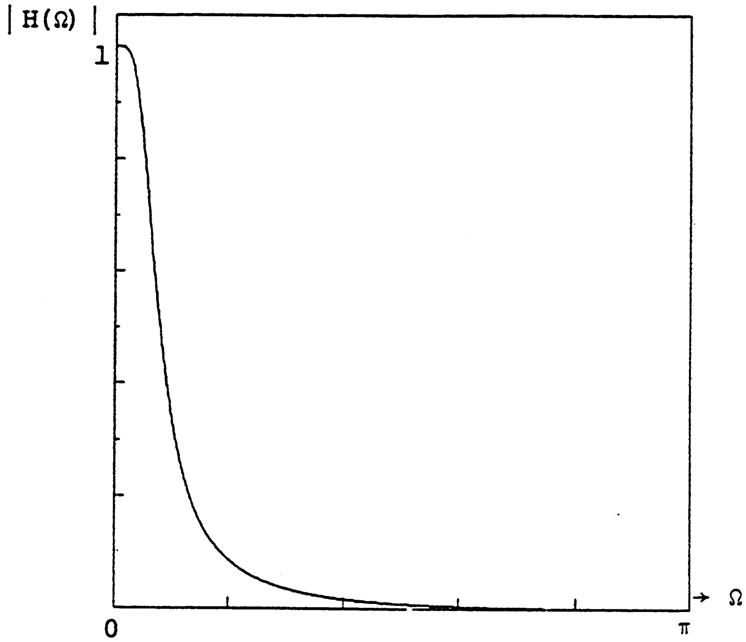
With $T = \dfrac{1}{10000}$,

$$H(z) = \frac{.0902z}{z^2 - 1.801z + .8189}$$

10.9 (a)

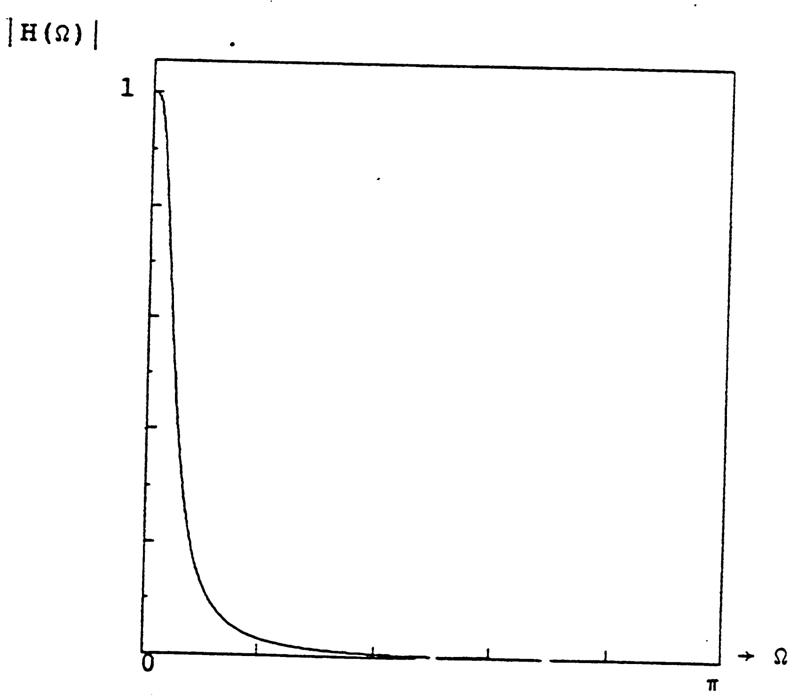


(b)



(b) Digital Filter with f_s = 6000 Hz
Problem 10.9

(c)



(c) Digital filter with f_s = 10 kHz
Problem 10.9

10.10 $\omega_p^d = \dfrac{2}{1/3000} \tan\left(\dfrac{1000 \times \frac{1}{3000}}{2}\right) = 1009.36$

$$\omega_s^d = \frac{2}{1/3000} \tan\left(\frac{3000 \times \frac{1}{3000}}{2}\right) = 3277.81$$

From Problem 10.1, $\delta_1 = .20567$, $\delta_2 = 0.31623$,

$N = 1.434$ Choose $N = 2$ and use Eq. (10.3.15)

to get $\omega_s = 1154.19$

$$\therefore H(\rho) = \frac{K}{\rho^2 + \sqrt{2}\,(1154.19)\rho + (1154.19)^2}$$

$$= \frac{K}{\rho^2 + 1632.27\,\rho + (1154.19)^2}$$

Set $\rho = \dfrac{2}{(1/3000)} \dfrac{1 - z^{-1}}{1 + z^{-1}}$ to get

$$H(z) = \frac{0.02836\,(z+1)^2}{z^2 - 1.471\,z + .5844}$$

10.11 From problem 10.3, the normalized low-pass filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1.}$$

For $f_s = 300$ Hz, the predistorted frequencies are

$$\omega_{c_1}^d = 600 \tan\left(\frac{200}{600}\right) = 207.75$$

$$\omega_{c_2}^d = 600 \tan\left(\frac{300}{600}\right) = 327.78$$

so that $\omega_0^d = \sqrt{\omega_{c_1}^d \cdot \omega_{c_2}^d}$

$$B.W. = \omega_{c_2}^d - \omega_{c_1}^d = 120.03$$

10.12 (a)

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega_n}$$

$$= \sum_{n=-\infty}^{-1} h(n)e^{-j\Omega n} + h(0) + \sum_{n=1}^{\infty} h(n)e^{-j\Omega n}$$

$$= h(0) + \sum_{n=1}^{\infty}[h(n)e^{-j\Omega n} + h(-n)e^{j\Omega n}]$$

If $h(n) = h(-n)$,

$$H(\Omega) = h(0) + \sum_{n=1}^{\infty} 2h(n)\cos(\Omega n)$$

and $H(\Omega)$ is purely real.

If $h(n) = -\,h(-n)$, $h(0) = 0$, so that

$$H(\Omega) = j\sum_{n=1}^{\infty}(-2)h(n)\sin(\Omega n)$$

and is purely imaginary.

(b) Let N be odd and let $g(n) = h(n + \frac{N-1}{2})$.

Then

$$G(\Omega) = e^{j(\frac{N-1}{2})\Omega}H(\Omega)$$

and

$$\mathrm{Arg}G(\Omega) = (\frac{N-1}{2})\Omega + \mathrm{Arg}H(\Omega)$$

If $h(n) = h(N-1-n)$, $g(n) = g(-n)$ and $\mathrm{Arg}G(\Omega) = 0$, so that

$$\mathrm{Arg}H(\Omega) = -\,(\frac{N-1}{2})\Omega$$

If $h(n) = -\,h(N-1-n)$, $g(n) = -\,g(-n)$ and $\mathrm{Arg}G(\Omega) = -\,\frac{\pi}{2}$, so that

$$\mathrm{Arg}H(\Omega) = -\,(\frac{N-1}{2})\Omega - \frac{\pi}{2}$$

10.13 (a)

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\Omega e^{j\Omega n}\,d\Omega$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \Omega\sin(\Omega n)\,d\Omega$$

$$= \begin{cases} 0 & n = 0 \\ \dfrac{(-1)^n}{n} & \text{otherwise} \end{cases}$$

(b) With an 11-point response and a delay of 6 samples, we have

$$h(n) = [\tfrac{1}{5}\ \ \tfrac{-1}{4}\ \ \tfrac{1}{3}\ \ \tfrac{-1}{2}\ \ 1\ \ 0\ \ -1\ \ \tfrac{1}{2}\ \ \tfrac{-1}{3}\ \ \tfrac{1}{4}\ \ \tfrac{-1}{5}]$$

so that

$$H(z) = \frac{1}{5}(1 - z^{-10}) - \frac{1}{4}(z^{-1} - z^{-9}) + \frac{1}{3}(z^{-2} - z^{-8}) - \frac{1}{2}(z^{-3} - z^{-7}) + (z^{-4} - z^{-6})$$

The 11-point Hanning window is

$$w_h(n) = [.067\ \ .25\ \ .5\ \ .75\ \ .933\ \ 1\ \ .933\ \ .75\ \ .5\ \ .25\ \ .067]$$

The corresponding impulse response of the differentiator is

$$h_h(n) = [.0134\ \ -.0625\ \ .1667\ \ -.375\ \ .933\ \ 0\ \ -.933\ \ .375\ \ -.1667\ \ .0625\ \ -.0134]$$

with transfer function

$$H_h(z) = 0.0134(1 - z^{-10}) - .0625(z^{-1} - z^{-9}) + 0.1667(z^{-2} - z^{-8})$$

$$-0.375(z^{-3} - z^{-7}) + 0.933(z^{-4} - z^{-6})$$

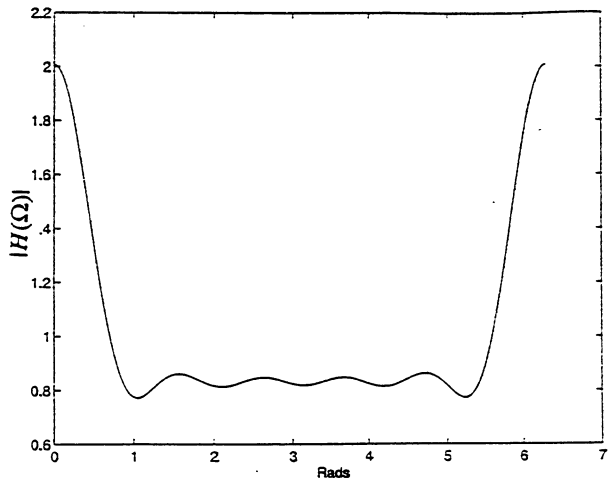
10.14 (a) The ideal filter has impulse response $h(n) = \sin(\pi n/6)/\pi n$. The 12-tap filter response with a delay of 6 is given by

$$h(n) = [0\ \ .0318\ \ .0689\ \ .1061\ \ .1378\ \ .1592\ \ 1\ \ .1592\ \ .1378\ \ .1061\ \ .0689\ \ .0318\ \ 0]$$

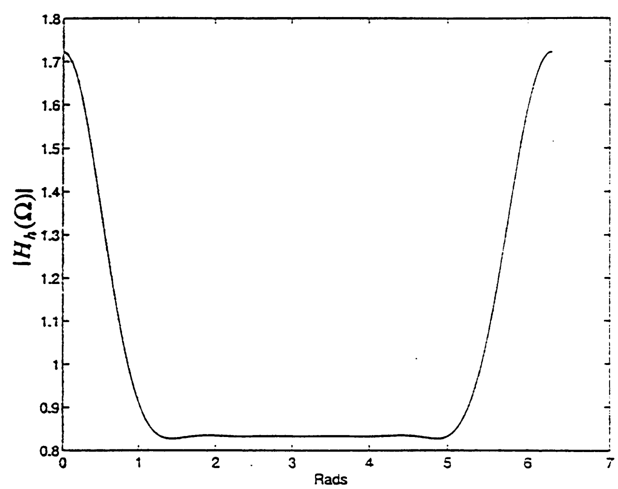
The impulse response of the filter with the Hanning window is

$$h_h(n) = [0\ \ .006\ \ .0268\ \ .0649\ \ .0119\ \ .1513\ \ 1\ \ .1513\ \ .0119\ \ .0649\ \ .0268\ \ .006\ \ 0]$$

Plots of $|H(\Omega)|$ and $|H_h(\Omega)|$ are as shown below. It is clear that a 12-tap filter does not provide a good approximation to the desired filter response.



2.2
2
1.8
1.6
.4
1.2
1
0.8
0.6
0
1
2
3
4
5
6
7
Rads
$|H(\Omega)|$



1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0
1
2
3
4
5
6
7
Rads
$|H_h(\Omega)|$